A sharp Liouville theorem and polynomial approximation of non-analytic functions in Orlicz spaces

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Abstract

We report on our current research project where we prove the following results: Let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function with $\varphi(0) = 0$. Then there exists a non-trivial entire function f such that

$$\int_{\mathbb{C}} \varphi(|f(z)|) dA(z) < \infty$$

if and only if

$$\int_0^\infty \frac{dx}{1 + \varphi(e^{e^x})} = \infty.$$

Furthermore, when this divergence condition holds, the space of entire functions is dense in the Orlicz space $L_\varphi(\mathbb{C})$. Similarly we prove that for any compact set $K \subset \mathbb{C}$ with non-empty interior, then the set of polynomials is dense in $L_\varphi(K)$ if and only if this integral diverges. Remarkably, this allows us to approximate non-analytic functions, such as $f(z) = \bar{z}$, by polynomials in this Orlicz space. In particular, our results resolves a gap left open by Kalton (1980) for the case where $\varphi(x) = (\log^+ \log^+ x)^p$ with $1 < p \leq 2$. Methods of proofs include a Carleman-type differential inequality and the tangential Arakelyan approximation theorem. *Note:* This is our first piece of research where we have used a large language model (Gemini Pro 2.5, 3.0, 3.1) as an assistant, helpful for brain storming, editing part of the text and finding relevant references.