ON HYPOTHESIS TESTING FOR IMPULSE RESPONSE FUNCTION

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Outline

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Impulse response function

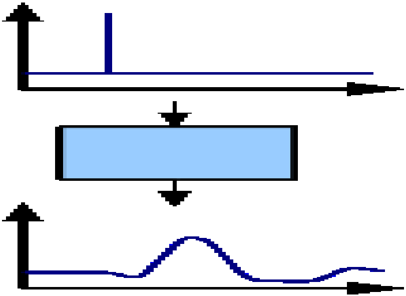

Figure: Some examples of Impulse response functions

Application

- Digital signal processing;
- Automatic control systems;
- Electronic processing;
- Econometrics (errors-in-variables models);
- Loudspeakers;
- Acoustic and audio applications;

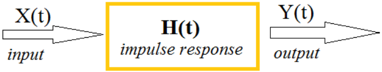

Figure: Structure of the system

Consider a time-invariant casual continuous linear system with response function $H(\tau)$, $\tau \in [0, \Lambda]$.
$H(\tau) = 0$ as $\tau < 0$,

$$Y(t) = \int_0^{\Lambda} H(\tau) X(t-\tau) d\tau. \tag{1}$$

? to estimate or identify the function H by observations.

Construct the estimator $\hat{H}_\Lambda(\tau)$ of H and

Find the accuracy

$$P\{\sup_{\tau\in[0,\Lambda]} |\hat{H}_\Lambda(\tau) - H(\tau)| \geq \varepsilon\}, \quad \varepsilon > 0.$$

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Under a_0 we define the output of the system on the constant signal. This means that

$$a_0 = \frac{1}{\sqrt{\Lambda}} \int_0^{\Lambda} H(t)dt. \tag{2}$$

Set

$$H^*(\tau) = H(\tau) - a_0. \tag{3}$$

As an estimator of the difference of impulse response function and a_0 $H^*(\tau)$ we will consider an integral cross-correlogram

$$\hat{H}(\tau) = \hat{H}_{T,\Lambda}(\tau) = \frac{1}{T} \int_0^T Y(t)X(t-\tau)dt, \quad \tau \in [0, \Lambda] \tag{4}$$

where $T > 0$ is a parameter for averaging.

Suppose now that the input signal processes of the LTI system are zero mean stationary Gaussian stochastic processes that are given by:

$$X_N(u) = \sqrt{\frac{2}{\Lambda}} \sum_{k=1}^{N} \left(\xi_k \cos(\frac{2k\pi u}{\Lambda}) + \eta_k \sin(\frac{2k\pi u}{\Lambda}) \right), \quad u \in [0, \Lambda] \tag{5}$$

Assumptions

Condition A. The function $H(\tau)$ is two times differentiable on $[0,\Lambda]$. The functions $H(\tau)$ and $H'(\tau)$ are continuous on $[0,\Lambda]$ and

$$
\begin{aligned}
I_0 &= I_0(\Lambda) = \int_0^{\Lambda} |H(\tau)| d\tau < \infty, \\
I_1 &= I_1(\Lambda) = \left(\int_0^{\Lambda} |H'(\tau)|^2 d\tau \right)^{1/2} < \infty, \\
I_2 &= I_2(\Lambda) = \int_0^{\Lambda} |H''(\tau)| d\tau < \infty.
\end{aligned}
$$

Condition B. The following relation holds true

$$H(0) = H(\Lambda).$$

Let's denote

$$d := |H'(0)| + |H'(\Lambda)|. \tag{6}$$

Lemma

*Assume that the conditions **A, B** are satisfied. Then*

$$|H^*(\tau) - \mathbf{E}\hat{H}_{N,T,\Lambda}(\tau)| \le \frac{\Lambda(d + l_2(\Lambda))}{2\pi^2 N}, \tag{7}$$

$$Var\,\hat{H}_{N,T,\Lambda}(\tau) \le \frac{\Lambda^3(\Lambda+2)l_1^2}{\pi^4 T^2}\left(2 - \frac{1}{N}\right)^2, \tag{8}$$

*where $l_1(\Lambda)$ and $l_2(\Lambda)$ are defined in condition **A**.*

The following lemma gives an estimation for the variance of difference $\hat{H}_{N,T,\Lambda}(\tau) - \hat{H}_{N,T,\Lambda}(\theta)$.

Lemma

Suppose that the conditions of Lemma 1 are fulfilled. Then

$$Var(\hat{H}_{N,T,\Lambda}(\tau) - \hat{H}_{N,T,\Lambda}(\theta)) \leq \tilde{C}(N,T,\Lambda)|\tau-\theta|^{\alpha}, \alpha \in (0,1],$$

where $\tilde{C}(N,T,\Lambda)$ is some constant and $\tau, \theta \in [0,\Lambda]$.

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Denote

$$h^*_{N,\Lambda} = \frac{\Lambda(d + I_2(\Lambda))}{2\pi^2 N}.$$

Then from (7) it follows that

$$|E\hat{H}_{N,T,\Lambda}(\tau) - H^*(\tau)| \leq h^*_{N,\Lambda}, \quad \tau \in [0, \Lambda].$$

Put

$$\gamma_0(N, T, \Lambda) = \gamma_0 = \frac{\Lambda\sqrt{\Lambda(\Lambda+2)}I_1}{\pi^2 T}\left(2 - \frac{1}{N}\right). \tag{9}$$

From (8) we have that

$$\sup_{\tau\in[0,\Lambda]} \sqrt{Var\hat{Z}_{N,T,\Lambda}(\tau)} \leq \gamma_0.$$

Let

$$M_\alpha = 2^{2-\frac{1}{2\alpha}} e^{1/\alpha} \gamma_0^{-\frac{1}{2}-\frac{1}{\alpha}} \alpha^{1/\alpha - 1/2}.$$

Theorem

Suppose that the conditions **A**, **B** *are fulfilled. Then the inequality*

$$P\left\{\sup_{\tau\in[0,\Lambda]}|H^*(\tau)-\hat{H}_{N,T,\Lambda}(\tau)|>\varepsilon\right\}\leq M_\alpha(\varepsilon-h^*_{N,\Lambda})^{\frac{1}{\alpha}}$$
$$\times\left(C\alpha+\sqrt{2}\alpha(\varepsilon-h^*_{N,\Lambda})-2\gamma_0\right)^{\frac{1}{2}}\exp\left\{-\frac{\varepsilon-h^*_{N,\Lambda}}{\sqrt{2}\gamma_0}+\frac{1}{\alpha}\right\} \quad (10)$$

holds true for

$$\varepsilon>\frac{\sqrt{2}\gamma_0}{\alpha}+h^*_{N,\Lambda},\quad \alpha\in(0,1].$$

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It is possible to test hypothesis on the shape of impulse response function.
$\mathbf{H_0}$ states that an impulse response function is $H(\tau)$, $\tau \in [0, \Lambda]$, and $\mathbf{H_a}$ implies the opposite statement.
Denote

$$\begin{aligned} g_1(\varepsilon) = g_1(\varepsilon, N, T) &= M_\alpha(\varepsilon - h^*_{N,\Lambda})^{\frac{1}{\alpha}} \left(C\alpha + \sqrt{2}\alpha(\varepsilon - h^*_{N,\Lambda}) - 2\gamma_0 \right)^{\frac{1}{2}} \\ &\times \exp\left\{ -\frac{\varepsilon - h^*_{N,\Lambda}}{\sqrt{2}\gamma_0} + \frac{1}{\alpha} \right\}. \qquad (11) \end{aligned}$$

From Theorem 1 follows that if
$\varepsilon > z_{N,T,\Lambda} = \frac{\sqrt{2}\gamma_0}{\alpha} + h^*_{N,\Lambda}, \quad \alpha \in (0, 1]$, then

$$P\left\{ \sup_{\tau \in [0,\Lambda]} |H(\tau) - \hat{H}_{N,T,\Lambda}(\tau)| > \varepsilon \right\} \le g_1(\varepsilon).$$

Let $\varepsilon_{1,\delta}$ be a solution of the equation

$$g_1(\varepsilon_{1,\delta}) = \delta, \quad 0 < \delta < 1.$$

δ is a significant level. Put

$$\varepsilon^*_{1,\delta} = \max\{\varepsilon_{1,\delta}, z_{N,T,\Lambda}\}. \tag{12}$$

It is obvious that $g_1(\varepsilon^*_{1,\delta}) \leq \delta$ and

$$P\left\{\sup_{\tau\in[0,\Lambda]} |H(\tau) - \hat{H}_{N,T,\Lambda}(\tau)| > \varepsilon^*_{1,\delta}\right\} \leq \delta.$$

From Theorem 1 it follows that to test the hypothesis $\mathbf{H_0}$, we can use the following criterion.

Criterion

For a given level of confidence $1-\delta$, $\delta \in (0,1)$, *the hypothesis* $\mathbf{H_0}$ *is rejected if*

$$\sup_{\tau\in[0,\Lambda]} |H(\tau) - \hat{H}_{N,T,\Lambda}(\tau)| > \varepsilon^*_{1,\delta},$$

otherwise the hypothesis $\mathbf{H_0}$ *is accepted, where* $\varepsilon^*_{1,\delta}$ *is from (12).*

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We consider a particular case when $\Lambda = 10$, $\alpha = 1$. An impulse response function is supposed to be

$$H(\tau) = \tau(e^{-\tau} - e^{-\Lambda}).$$

$$\begin{aligned} I_0 &= 0.9972, & I_1 &= 0.4998, \\ I_2 &= 1.2702, & d &= 1.0004. \end{aligned}$$

The function $H^*(\tau) = H(\tau) - a_0$ is equal to

$$H^*(\tau) = \tau(e^{-\tau} - e^{-\Lambda}) - 0.3153.$$

Table: The minimal values of T for fixed value of $N = 100$ with given significant level δ and accuracy ε for $g_1(\varepsilon, N) = \delta$

Accuracy, ε	T when $\delta = 0.3$	T when $\delta = 0.2$	T when $\delta = 0.1$
0.3	683 (0.0253)	725 (0.0239)	796 (0.0217)
0.5	404 (0.0429)	428 (0.0405)	471 (0.0368)
1	199 (0.0871)	212 (0.0817)	233 (0.0743)
1.5	136 (0.1274)	145 (0.1195)	159 (0.1090)
2	98 (0.1768)	105 (0.1650)	116 (0.1494)
5	40 (0.4333)	42 (0.4127)	46 (0.3768)

Outline

European Women in Mathematics

European Women in Mathematics (EWM) is an association of female mathematicians. The Association is involved in policy and strategic work in promoting the role of women in mathematics and offers direct support to its members. EWM was founded in 1986 and has achieved a membership of around 400.

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Thank your for listening!