Response set imbalance and non-response bias: a theoretical study with full use of auxiliary information

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The crucial question: Will better balanced response guarantee better accuracy (lower variance and/or bias) in the estimates?

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and a **design weight** $d_k = 1/\pi_k$.

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It is assumed that we have access to **auxiliary variables** $\mathbf{x}_k = (x_{k1}, x_{k2}, ..., x_{kJ})'$ that are known $\forall k \in s$ and we know the population totals $\mathbf{X} = \sum_U \mathbf{x}_k$.

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We assume that the auxiliary vector can be constructed as such that

$$\boldsymbol{\mu}'\mathbf{x}_k = 1, \forall k \in s, \text{for some vector } \boldsymbol{\mu} \text{ independent on } k.$$

Balance and imbalance

The response set is **balanced** if

$$\bar{\mathbf{x}}_r = \frac{\sum_r d_k \mathbf{x}_k}{\sum_r d_k} = \frac{\sum_s d_k \mathbf{x}_k}{\sum_s d_k} = \bar{\mathbf{x}}_s.$$

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Guiding data collection with *IMB* - **monitoring response**.

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Weights w_k satisfy calibration requirements:

$$\sum_{s} d_k w_k \mathbf{x}_k' = \sum_{U} \mathbf{x}_k'.$$

Estimation under non-response

The expansion estimator:

$$\hat{Y}_{EXP} = \hat{N} \sum_r d_k y_k / \sum_r d_k = \hat{N} \bar{y}_r,$$

where $\bar{y}_r = \sum_r d_k y_k / \sum_r d_k$.

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The calibration estimator under non-response:

$$\hat{Y}_{CAL} = \sum_r d_k g_k y_k,$$

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Imbalance of study variable

The study variable imbalance is characterised by

$$\bar{y}_r - \bar{y}_s,$$

where $\bar{y}_s = \sum_s d_k y_k / \sum_s d_k$ and $\bar{y}_r = \sum_r d_k y_k / \sum_r d_k$.

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$$\hat{N}\left(\bar{y}_r - \bar{y}_s\right) = \hat{Y}_{EXP} - \hat{Y}_{FUL}.$$

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$$\hat{N}\,(\bar{y}_r - \bar{y}_s) = \hat{Y}_{EXP} - \hat{Y}_{FUL}.$$

Let us expand the right side by $\pm\hat{Y}_{CAL}$:

$$\hat{N}\,(\bar{y}_r - \bar{y}_s) \;=\; \left(\hat{Y}_{EXP} - \hat{Y}_{CAL}\right) + \left(\hat{Y}_{CAL} - \hat{Y}_{FUL}\right)$$

Non-response bias

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Non-response bias

$$\begin{aligned}\hat{N}(\bar{y}_r - \bar{y}_s) &= \left(\hat{Y}_{EXP} - \hat{Y}_{CAL}\right) + \left(\hat{Y}_{CAL} - \hat{Y}_{FUL}\right) \\ &= \hat{N}(\bar{\mathbf{x}}_r - \bar{\mathbf{x}}_s)' \mathbf{b}_r + \hat{N}(\mathbf{b}_r - \mathbf{b}_s)' \bar{\mathbf{x}}_s.\end{aligned}$$

where $\mathbf{b}_r = \left(\sum_r d_k \mathbf{x}_k \mathbf{x}_k'\right)^{-1} \sum_r d_k \mathbf{x}_k y_k$ and
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This decompositions highlights two undesirable differences:

- Difference due to imbalance in the response

- Difference due to biased regression

Previous results

Let's denote

$$\Delta_r = (\mathbf{b}_r - \mathbf{b}_s)' \, \bar{\mathbf{x}}_s = \left(\hat{Y}_{CAL} - \hat{Y}_{FUL} \right) / \hat{N}$$

and investigate the effect of imbalance on Δ_r.

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Särndal et. al (2016) showed that under certain simplifying conditions, the conditional mean $E(\Delta_r|\bar{\mathbf{x}}_r, m, s) = 0$ and the conditional variance

$$V\left(\Delta_r|\bar{\mathbf{x}}_r, m, s\right) \approx \frac{S_y^2}{m}\left(1 - p + \frac{IMB}{p^2}\right)$$

where m is the number of respondents, $p = m/n$ is the response rate, $S_y^2 = \sum_{j=1}^{J} n_j/nS_{yj}^2$ and $S_{yj}^2 = \sum_{s_j}(y_k - \bar{y}_{s_j})^2/(n_j - 1), j = 1, ..., J$.

Further exploration

For simplification let us redefine the calibration estimator under non-response:

$$\hat{Y}_{CAL2} = \sum_r d_k g_{Uk} y_k,$$

where $g_{Uk} = \left(\sum_U \mathbf{x}_k\right)' \left(\sum_r d_k \mathbf{x}_k \mathbf{x}_k'\right)^{-1} \mathbf{x}_k$.

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Let us expand the $(\bar{y}_r - \bar{y}_s)$ decomposition with $\pm \hat{Y}^*_{CAL}$:

$$\hat{N}\left(\bar{y}_r - \bar{y}_s\right) = \left(\hat{Y}_{EXP} - \hat{Y}_{CAL2}\right) + \left(\hat{Y}_{CAL2} - \hat{Y}^*_{CAL}\right) + \left(\hat{Y}^*_{CAL} - \hat{Y}_{FUL}\right)$$

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$$\begin{aligned}
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&= \hat{N}\left(\bar{\mathbf{x}}_r - \bar{\mathbf{x}}_U\right)' \mathbf{b}_r + \hat{N}\left(\mathbf{b}_r - \mathbf{b}_s\right)' \bar{\mathbf{x}}_U + \hat{N}\left(\bar{\mathbf{x}}_U - \bar{\mathbf{x}}_s\right)' \mathbf{b}_s,
\end{aligned}$$

where $\bar{\mathbf{x}}_U = \sum_U \mathbf{x}_k / \sum_s d_k$.

Further exploration

Let the auxiliary vector be a grouping vector, so that $\mathbf{x}_k = (0, \dots, 1, \dots, 0)'$, where the only 1 indicates the unique group (out of J possible) to which k belongs. Then

$$\hat{N}\left(\bar{\mathbf{x}}_r - \bar{\mathbf{x}}_U\right)' \mathbf{b}_r = \hat{N} \sum_{j=1}^{J} \bar{y}_{rj} \left(\frac{m_j}{m} - \frac{N_j}{\hat{N}} \right),$$

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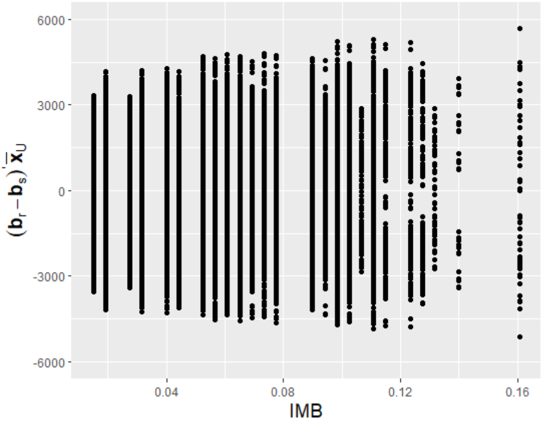
$$\hat{N}\left(\bar{\mathbf{x}}_U - \bar{\mathbf{x}}_s\right)' \mathbf{b}_s = \hat{N}\sum_{j=1}^{J} \bar{y}_{sj}\left(\frac{N_j}{\hat{N}} - \frac{n_j}{n}\right).$$

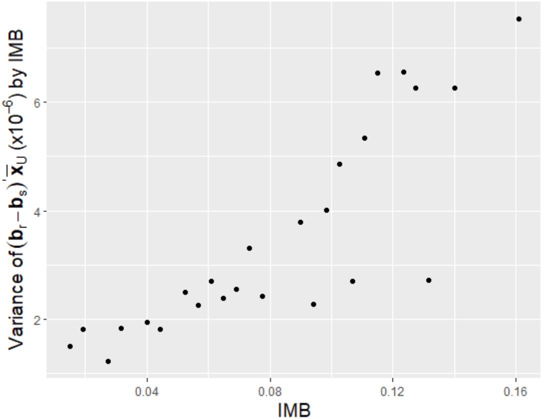
Simulations

A sample of $n = 20$ is fixed and all possible response sets are considered where $m = 12$. The auxiliary vector is a group vector, *IMB* and $(\mathbf{b}_r - \mathbf{b}_s)' \bar{\mathbf{x}}_U$ is calculated for 56 576 response sets.



Simulations

Variance of $(\mathbf{b}_r - \mathbf{b}_s)' \bar{\mathbf{x}}_U$ by *IMB* value.



References:

- Särndal, C.E., Lumiste, K., and Traat, I. (2016) Reducing the Response Imbalance: Is the Accuracy of the Survey Estimates Improved? *Survey Methodology*, 42 (2): 219–238.
- Lumiste, K. (2018) *Improving accuracy of survey estimators by using auxiliary information in data collection and estimation stages.* Dissertation, University of Tartu.