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Nonresponse bias for some common estimators and its change over time in the data collection process

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Abstract

In most surveys, the risk of nonresponse is a factor taken into account at the planning stage. Commonly, resources are set aside for a follow-up procedure which aims at reducing the nonresponse rate. However, we should pay attention to the effect of nonresponse, rather than the nonresponse rate itself. When considering nonresponse error, i.e. bias and variance, it is not obvious that the resources spent on nonresponse rate reduction efforts are time and money well spent. In this paper we address this issue, focusing on the effect of follow-ups on nonresponse bias. The nonresponse biases for some common estimators are derived, and the change in bias for these estimators is studied under a setup that allows us to take into account the data collection process, and follow-up efforts in particular.

Contents

1 Introduction

Nonresponse is a reality in every survey today, and is also an increasing problem in many countries. More and more efforts and resources are devoted to follow-ups, tracing and refusal conversion. Measures to reduce the nonresponse rate are considered a necessary part of surveys and special procedures are indeed usually required to keep the nonresponse rate low. A low nonresponse rate has been, and often is, regarded as essential to prevent the estimates from "having too much non-response bias". It is common to report the non-response rate as an indicator of the size of the nonresponse error. However, a low nonresponse rate is not necessarily a safeguard against severe bias, which means that the resources spent on increasing the response rate may not always be money well spent. In addition, the methods to reduce nonresponse are often used in a standardized fashion, making them inefficient.

Methods to deal with nonresponse can, roughly, be of three types: preventive, reductive and adjustive. The line between them is not clear-cut, especially not in interviewer-assisted surveys, but generally preventive actions are taken before the data collection begins, reductive efforts are used during the data collection stage, and adjustment is done at the estimation stage, after the data collection period is over. One method that combines elements of all three types is subsampling of nonrespondents, a method whereby unbiased estimation is guaranteed if full response can be achieved in a random subsample of nonrespondents in the original sample. The method was introduced by Hansen and Hurwitz (1946) and has been widely used.

Preventive methods include e.g. construction (and testing) of a questionnaire, interviewer recruiting and training and overall planning of the data collection. Another example is construction of a sampling frame with small overcoverage, since the risk of nonobservation may be larger for overcoverage elements than for population elements. Overcoverage then remains undetected and is often treated as belonging to the target population.

How the preventive tasks are performed can have a great impact on the response rate. The motivation for preventive efforts is however not only based on response rates, but should be done for many other reasons since they influence other types of errors as well. Preventive methods are usually independent of the choice of data collection mode, although the choice of an appropriate mode can in itself be regarded as a preventive action. Methods to reduce the nonresponse include e.g. mailing of reminders in postal surveys, repeated contact attempts in telephone surveys, tracing of respondents and the use of incentives. The choice of reductive measures is mainly governed by the data collection mode. It would for example be difficult to work with refusal conversion in a postal survey.

Most reductive and preventive methods have been developed from theories in behavioral science. In contrast, methods to adjust for nonresponse once it has occurred, are based on statistical theory. Adjustive methods include different reweighting approaches such as weighting by response homogeneity groups, calibration for nonresponse and imputation techniques. A combination of methods is frequently used, e.g. imputation for item missing data and reweighting for unit nonresponse. It is likely that the response mechanisms generating item and unit nonresponse are different, motivating different treatment of them at the estimation stage.

As one of the most reported non-sampling errors, non-response and its potentially harmful effects on survey estimates have undergone extensive research. Much of the research in recent years have been devoted to either methods for on the one hand non-response rate reduction and prevention and on the other hand adjustment techniques based on auxiliary information. Can something be said about the balance between the two?

In practice, there are budget constraints that must be met under quality requirements (high precision, small bias). The survey designer must allocate resources so that they are spent where they can be expected to yield the greatest error reduction. Nonresponse rate reduction efforts are generally expensive activities and constitute a large part of the available, limited, survey budget. For the cost of a reductive measure to be warranted, it should correspond to an increase in quality, in terms of reduced bias.

An overall goal would be to develop methodology for balancing nonresponse reduction strategies under budget constraints. We can, however, not hope to find one single, optimal, solution and indeed we do not even hope to formulate the problem in such a way that an overall optimal solution can be found. The practical use of cost and error models, which by necessity are (sometimes considerable) simplifications of the "real" problem, can be questioned. Groves (1989) discusses such criticisms presented by Fellegi and Sunter (1974). He points out that in a multipurpose survey the problem is normally so complex that no one optimal solution can be identified, but error modeling can be used as a tool with which different alternative strategies can be evaluated and compared.

As a first step, we will study the effect on estimator bias of reductive measures, such as callbacks or simply extension of the data collection period. All preventive efforts are assumed fixed. The issue here is whether or not the efforts and resources spent on reducing the nonresponse rate really leads to a worthwhile reduction of the nonresponse bias. The basic idea is that if it can be concluded that some follow-up efforts do not increase quality sufficiently, they can be left out, saving both money and time. Unfortunately, the effect of a single follow-up effort can not always be singled out, so the data collection procedure must be evaluated as a whole. For a complete evaluation of the effectiveness of the follow-ups, focus must also lie on the cost of the current follow-up procedures, how estimator variances change as response probabilities change as a result of reduction efforts and the possibility of estimating changes in bias and variance. Attempts along these lines will be made in future work, the present paper is the first in a project on balancing nonresponse reduction efforts and survey cost.

In considering a balance between reducing the nonresponse and adjusting for it, it becomes essential to regard the response probabilities not as fixed, but rather as variable, subject to influence from the survey organizers through follow-ups etc. A response distribution taking this into account is presented in section 2.1 and will be used to evaluate estimator properties (nonresponse bias).

As the purpose is to evaluate the follow-up procedure, focusing on its ability to reduce nonresponse bias, the interest lies not in the nonresponse bias itself but rather in if and how that bias changes during the data collection process. Even so, it is important to keep in mind that if the bias does not change between two points in time, this does not necessarily mean that the bias is low. It could be taken as evidence that the follow-up procedure is inefficient and consequently a waste of resources. However, the situation is more complex as there may be interaction effects between efforts. This effect is difficult to estimate without the use of embedded experiments. If it is found that the reduction efforts do not seem to reduce bias, the solution is not simply to exclude the last reminders and accept higher nonresponse, but rather to find alternative, more efficient follow-up procedures that can help reduce the bias substantially. Even though the follow-up procedure in its current form does not seem to *reduce* nonresponse error, this can not be taken as evidence that there is no error present. It is still essential that the magnitude of the bias is estimated.

Other approaches along these lines could be considered. Selective follow-ups could be introduced as part of the reduction strategy, i.e. targeting subpopulations where response probabilities are particularly low, or that are believed to be major sources of bias in estimators. Register data can be used to identify such subpopulations. This is a common strategy in business surveys, where a few businesses have a great impact on estimates. However, it is usually not applied systematically and the effect on bias is unknown. The effect on estimators of such a strategy could be evaluated using the framework presented here.

2 Notation, definitions and assumptions

2.1 The response distribution

Traditionally, the outlook on nonresponse has been either deterministic or probabilistic. A common and reasonable perspective is that there exists a true, but unknown, response mechanism. Since the response distribution, RD, is unknown, the evaluation of the estimator properties must be based on an assumed response distribution. Here it is assumed that whether an element responds to the survey request is a stochastic event.

One way to deal with the nonresponse at the estimation stage is to assume some response model and apply theory for two-phase sampling with estimated response probabilities as second phase inclusion probabilities. Hopefully the response model will correspond closely to the underlying response mechanism, in which case the nonresponse bias can be eliminated or at least greatly reduced. There are other ways to deal with the nonresponse, but the true response distribution will nevertheless be a determining factor for the properties of the estimator.

It is reasonable to think that the response distribution is influenced by the general survey conditions, such as the survey topic and the population's attitude to survey participation, but also by strategies to deal with nonresponse, and other, more specific aspects of the survey setting. The settings in a certain survey generates a specific response distribution with individual response probabilities that would most likely be different if the survey settings changed.

The response probabilities would, in a two-phase approach, be estimated at the estimation stage. However, to take into account and evaluate the effect of events taking place during the data collection process, a more elaborate response distribution must be introduced. We will assume that there is not one true response distribution (RD), fixed for elements in the sample, but rather that the response probabilities are influenced by the nonresponse reductive measures taken by the survey administrator and thus change during the process of data collection.

Consider a general procedure that consists of all the survey operations right up to, but not including, initial (attempted) contact. This includes all the general survey conditions and all preventive measures, generating a (fictitious) level of "response propensity". Given this procedure, reductive measures are taken with the purpose of increasing the (final) response probabilities. Unlike the general procedure, the reduction measures are applied only to the nonresponding subset of the sample at each given time.

The entire survey process right up to time a, including attempts at nonresponse reduction, induces $RD^{(a)}$. We assume that each additional effort induces larger response probabilities, at least for some elements in the sample, and that elements respond independently.

2.2 Notation and definitions

Consider a finite population $U = \{1, \ldots, k, \ldots, N\}$ consisting of N elements. Let s be a random sample of size n from U, selected according to the design $p(\cdot)$, with positive first and second order inclusion probabilities π_k and π_{kl}. We assume that direct element sampling without replacement is used.

The study variable is denoted y and the unknown value for the kth element is y_k. The parameter of interest is the population total

$$t_y = \sum_{k \in U} y_k = \textstyle\sum_U y_k$$

Sometimes it will be convenient to simplify the notation of estimators by using ˇ to represent division by π_k, e.g. $\check{y}_k = y_k/\pi_k$.

The data collection starts with an initial attempted contact (e.g. by mail or telephone) with all elements in the sample. Combined with the nonresponse reduction efforts, this yields successive response sets $r^{(1)} \subset \ldots \subset r^{(a)} \subset \ldots \subset r^{(A)} \subset s$, where a denotes some point of time during the data collection period and $r^{(A)}$ is the final response set in the current survey setup. Between time $a-1$ and a, the number of respondents increase, and the increase is the number of elements in the set $r^{(a)} \backslash r^{(a-1)}$. The response set $r^{(a)}$ is assumed to have been generated according to the induced response distribution $RD^{(a)}$, with response probabilities $\theta_{k|s}^{(a)}$.

The number of elements in response set $r^{(a)}$ will be denoted $m^{(a)}$. Since we consider the case of a single study variable, we need not make a distinction between item and unit nonresponse. When the data collection is completed at time a, estimation of t_y will be based on the response set $r^{(a)}$.

Remark 1 *The superscript (a) will be used throughout to denote estimators and quantities based on response set* $r^{(a)}$*.*

Also, let $\mathbf{x}_{1k}$ and $\mathbf{x}_{2k}$ denote the value for element k of two auxiliary vectors. We assume that

- $\sum_U \mathbf{x}_{1k}$ is known
- The vectors $\mathbf{x}_{1k}$ and $\mathbf{x}_{2k}$ are known for every $k \in s$

The available auxiliary information for $k \in s$ is thus $\mathbf{x}_k = (\mathbf{x}'_{1k}, \mathbf{x}'_{2k})'$.

3 Estimators adjusting for nonresponse

There are an abundance of methods to adjust for nonresponse at the estimation stage. Two main groups are reweighting and imputation. One frequently utilized method, combining the two, is imputation for item nonresponse followed by reweighting for unit nonresponse. Since we consider a single study variable, only reweighting methods will be considered here. Imputation can sometimes be a better alternative than reweighting, but will not be studied here. There is such a diversity in imputation methods; several methods can be applied for the same study variable with individual treatment of some elements (e.g. by expert opinion), making it difficult to apply general theory in situations where imputation is motivated.

The stochastic view of response discussed in section 2.1 has led to the use of two-phase sampling methods for dealing with nonresponse at the estimation stage. By making appropriate assumptions about the "selection" probabilities in phase two, standard estimation methods for "proper" two-phase samples can be applied. These methods require the formulation of explicit model assumptions about the unknown elicited response probabilities. One such model is the *response homogeneity groups* (RHG) model. Using the RHG model, regression estimators for two-phase samples can easily be adapted to the nonresponse situation.

In the calibration for nonresponse approach, response probabilities never actually enter the estimation formulas, but are implicit, resulting from the way the auxiliary information is applied in the estimator.

Both the calibration and the regression approaches actually generate classes of estimators, so that many alternative estimators are possible. The ones chosen here are believed to represent estimators most used in practice, although the selection is somewhat arbitrary. Since the main purpose is not to compare different estimators, but to study how a given estimator's properties change through the data collection, this should be of minor importance.

3.1 The response homogeneity groups model

The use of response homogeneity groups to adjust for nonresponse is motivated by the assumption that the true response probabilities are (approximately) equal within the groups. The response model is formulated given the sample and the general formulation is as follows: the realized sample s can be divided into groups s_h, $h = 1, \ldots, H_s$, such that response probabilities are constant within groups, but are not necessarily the same in different groups. The grouping need not be the same for different samples, hence the subscript s in H_s. We also assume that the response probabilities are positive for all k and that elements k and l respond independently. For an estimator based on the RHG model, the nonresponse bias is zero if the response model agrees with the true response distribution. Of course, one does not actually believe that the model holds exactly, only that it describes the response behavior sufficiently well to reduce nonresponse bias compared to more simple models.

We will assume that the same response model is used for any given time a. To keep in line with the notation already introduced, we can state the model formally as

$$\begin{aligned} Pr(k \in r^{(a)}|s) &= \theta^{(a)}_{k|s} = \theta^{(a)}_{hs} > 0 \text{ for all } k \in s_h \\ Pr(k\&l \in r^{(a)}|s) &= \theta^{(a)}_{k|s}\theta^{(a)}_{l|s} \text{ for all } k \neq l \in s \end{aligned}$$

for $h = 1, \ldots, H_s$ and $a = 1, \ldots, A$.

Let n_h be the size of s_h and let $r^{(a)}_h$ be the responding subset of s_h at time a. To form an estimator, estimated conditional response probabilities are used as weights. The conditioning is on the response count vector $\mathbf{m}^{(a)} = (m^{(a)}_1, \ldots, m^{(a)}_h, \ldots, m^{(a)}_{H_s})$, where $m^{(a)}_h$ is the size of $r^{(a)}_h$, i.e. the number of responding elements in s_h. Under the RHG model, the conditional response probabilities are estimated by

$$\hat{\theta}^{(a)}_{k|s,\mathbf{m}^{(a)}} = \hat{\theta}^{(a)}_{hs} = \frac{m_h^{(a)}}{n_h} \text{ for all } k \in s_h$$

and

$$\hat{\theta}^{(a)}_{kl|s,\mathbf{m}^{(a)}} = \begin{cases} \dfrac{m_h^{(a)}(m_h^{(a)}-1)}{n_h(n_h-1)} & \text{för } k \neq l \in s_h \\ \dfrac{m_h^{(a)}}{n_h}\dfrac{m_{h'}^{(a)}}{n_{h'}} & \text{för } k \in s_h, l \in s_{h'}; h \neq h' \end{cases}$$

We will study four different RHG estimators, one where auxiliary information is used only to form the response homogeneity groups, and three different regression based RHG estimators, one general and two special cases, with auxiliary information available at different levels.

The first estimator is the simple RHG estimator

$$\hat{t}^{(a)}_{yc\pi^*} = \textstyle\sum_{r^{(a)}} \dfrac{y_k}{\pi_k \hat{\theta}^{(a)}_{k|s}} = \displaystyle\sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \textstyle\sum_{r_h^{(a)}} \check{y}_k \tag{1}$$

The second estimator is the regression estimator in the general case of auxiliary information both for the population ($\mathbf{x}_1$) and for the sample ($\mathbf{x}$)

$$\hat{t}^{(a)}_{ycreg} = \textstyle\sum_U \hat{y}^{(a)}_{1k} + \sum_s \dfrac{\hat{y}^{(a)}_k - \hat{y}^{(a)}_{1k}}{\pi_k} + \displaystyle\sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \textstyle\sum_{r_h^{(a)}} \dfrac{y_k - \hat{y}^{(a)}_k}{\pi_k} \tag{2}$$

which is a sum of predicted values and weighted residuals, with predictions

$$\hat{y}^{(a)}_{1k} = \mathbf{x}'_{1k}\hat{\mathbf{B}}^{(a)}_{1r} = \mathbf{x}'_{1k}\left(\sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \textstyle\sum_{r_h^{(a)}} \dfrac{\mathbf{x}_{1k}\mathbf{x}'_{1k}}{\sigma^2_{1k}\pi_k}\right)^{-1} \displaystyle\sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \textstyle\sum_{r_h^{(a)}} \dfrac{\mathbf{x}_{1k} y_k}{\sigma^2_{1k}\pi_k}$$

and

$$\hat{y}^{(a)}_{k} = \mathbf{x}'_{k}\hat{\mathbf{B}}^{(a)}_{r} = \mathbf{x}'_{k}\left(\sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \textstyle\sum_{r_h^{(a)}} \dfrac{\mathbf{x}_{k}\mathbf{x}'_{k}}{\sigma^2_{k}\pi_k}\right)^{-1} \left(\displaystyle\sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \textstyle\sum_{r_h^{(a)}} \dfrac{\mathbf{x}_{k} y_k}{\sigma^2_{k}\pi_k}\right)$$

Here, σ^2_{1k} and σ^2_k are residual variances in an assumed, hypothetical regression model of y on $\mathbf{x}_1$ and $\mathbf{x}$, respectively.

The two special cases of this estimator that we will study are

$$\hat{t}_{ycreg1}^{(a)} = \sum_U \hat{y}_{1k}^{(a)} + \sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \sum_{r_h^{(a)}} \frac{y_k - \hat{y}_{1k}^{(a)}}{\pi_k} \tag{3}$$

which uses only $\mathbf{x}_{1k}$, the auxiliary information available for $k \in U$, and

$$\hat{t}_{ycreg2}^{(a)} = \sum_s \frac{\hat{y}_k^{(a)}}{\pi_k} + \sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \sum_{r_h^{(a)}} \frac{y_k - \hat{y}_k^{(a)}}{\pi_k} \tag{4}$$

that is used when $\mathbf{x}_k$ is the auxiliary information, available only for $k \in s$.

All four RHG estimators are taken from Särndal, Swensson, and Wretman (1992), who show their unbiasedness if the RHG model coincides with the true response distribution, under the condition that there is a negligible probability of one or more empty groups. They also present (approximate) variances and variance estimators, valid under the assumption that the response model holds.

Remark 2 *The regression estimators can alternatively be expressed as g-weighted sums over the response set. In the general case we have*

$$\hat{t}_{ycreg}^{(a)} = \sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \sum_{r_h^{(a)}} \left[g_{1kr}^{*(a)} + g_{2kr}^{(a)} - 1\right] \frac{y_k}{\pi_k} = \sum_{r^{(a)}} \left[g_{1kr}^{*(a)} + g_{2kr}^{(a)} - 1\right] \frac{y_k}{\pi_k \hat{\theta}_{k|s}^{(a)}}$$

with

$$g_{1kr}^{*(a)} = 1 + \left(\sum_U \mathbf{x}_{1k} - \sum_s \frac{\mathbf{x}_{1k}}{\pi_k}\right)' \left(\sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \sum_{r_h^{(a)}} \frac{\mathbf{x}_{1k}\mathbf{x}_{1k}'}{\sigma_{1k}^2 \pi_k}\right)^{-1} \frac{\mathbf{x}_{1k}}{\sigma_{1k}^2}$$

and

$$g_{2kr}^{(a)} = 1 + \left(\sum_s \frac{\mathbf{x}_k}{\pi_k} - \sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \sum_{r_h^{(a)}} \frac{\mathbf{x}_k}{\pi_k}\right)' \left(\sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \sum_{r_h^{(a)}} \frac{\mathbf{x}_k\mathbf{x}_k'}{\sigma_k^2 \pi_k}\right)^{-1} \frac{\mathbf{x}_k}{\sigma_k^2}$$

The estimators in the two special cases can be written

$$\hat{t}_{ycreg1}^{(a)} = \sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \sum_{r_h^{(a)}} g_{1kr}^{(a)} \frac{y_k}{\pi_k} = \sum_{r^{(a)}} g_{1kr}^{(a)} \frac{y_k}{\hat{\theta}_{k|s}^{(a)} \pi_k}$$

with

$$g_{1kr}^{(a)} = 1 + \left(\sum_U \mathbf{x}_{1k} - \sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \sum_{r_h^{(a)}} \frac{\mathbf{x}_{1k}}{\pi_k}\right)' \left(\sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \sum_{r_h^{(a)}} \frac{\mathbf{x}_{1k}\mathbf{x}_{1k}'}{\sigma_{1k}^2 \pi_k}\right)^{-1} \frac{\mathbf{x}_{1k}}{\sigma_{1k}^2}$$

and

$$\hat{t}_{ycreg2}^{(a)} = \sum_{h=1}^{H_s} \frac{n_h}{m_h^{(a)}} \sum_{r_h^{(a)}} g_{2kr}^{(a)} \frac{y_k}{\pi_k} = \sum_r g_{2kr}^{(a)} \frac{y_k}{\pi_k \hat{\theta}_{k|s}^{(a)}}$$

3.2 Calibration for nonresponse

The method of calibration for nonresponse originated from the notion that two-phase methods with response probability modeling are unnecessarily complex, and that practicing survey statisticians would benefit from a (computationally) "simpler" method. It is an extension of the calibration theory in a situation with full response, introduced by Deville and Särndal (1992). In calibration, the design weights $1/\pi_k$ form starting weights from which new, calibrated weights are calculated. These calibration weights lie as close as possible to the starting weights, but still produce known quantities when applied to the auxiliary vector. Originally, the calibration for nonresponse approach was developed for situations where auxiliary information was available either for the population or for the sample, not for both. In those cases, the calibration weights are calculated in a single computational step, and no modeling of the response probabilities (or the study variable) is necessary. The methods can be generalized to the case when $\mathbf{x}_1$ and $\mathbf{x}_2$ are used simultaneously, but the situation becomes more complex. Suggestions in this direction have been made by Särndal, drawing on calibration methods for two-phase samples (Särndal, Lecture notes, 2003). This case will not be studied, since the properties of the estimator have not been thoroughly studied, and it is not widely used. A brief description is given in remark 3.

The success of the calibration approach relies entirely on the strength of the auxiliary information, and how it is being used. The method may seem simpler since it does not require any modeling, but it still requires a careful selection of an appropriate auxiliary vector.

The estimators in the two special cases that will be studied here can both be written on the form

$$\hat{t}_{ycal}^{(a)} = \sum_{r^{(a)}} v_{kr}^{(a)} \frac{y_k}{\pi_k} \tag{5}$$

with

$$v_{kr}^{(a)} = v_{1kr}^{(a)} = 1 + \left(\sum_U \mathbf{x}_{1k} - \sum_{r^{(a)}} \frac{\mathbf{x}_{1k}}{\pi_k} \right)' \left(\sum_{r^{(a)}} \frac{c_k \mathbf{x}_{1k} \mathbf{x}'_{1k}}{\pi_k} \right)^{-1} c_k \mathbf{x}_{1k} \quad (6)$$

in special case 1 and

$$v_{kr}^{(a)} = v_{2kr}^{(a)} = 1 + \left(\sum_s \frac{\mathbf{x}_k}{\pi_k} - \sum_{r^{(a)}} \frac{\mathbf{x}_k}{\pi_k} \right)' \left(\sum_{r^{(a)}} \frac{c_k \mathbf{x}_k \mathbf{x}'_k}{\pi_k} \right)^{-1} c_k \mathbf{x}_k \quad (7)$$

in special case 2. The factors c_k are specified by the statistician. In the first case, we have calibrated on $\mathbf{x}_1$ from $r^{(a)}$ to U, in the second on $\mathbf{x}$ from $r^{(a)}$ to s.

The estimators are given in Lundström (1997).

Remark 3 *As indicated above, it would be possible to form a calibration estimator in the general case, i.e. when we have two auxiliary vectors, one with the population sum known, and one available only for sample elements. In that case, there are alternative ways to calculate the calibrated weights. It is suggested that the starting weights are chosen as the product of the design weights and a simple expansion factor* $F_k \geq 1$ *that will compensate for the nonresponse. How the choice of* F_k *affects the nonresponse bias has not yet been investigated.*

In one-step calibration, final weights v_k *are calculated, based on starting weights* $\frac{1}{\pi_k} F_k$*, so that the calibration equation* $\sum_r v_k \mathbf{x}_k = \begin{pmatrix} \sum_U \mathbf{x}_{1k} \\ \sum_s \mathbf{x}_{2k}/\pi_k \end{pmatrix}$ *is fulfilled. The final weights are then*

$$v_k = \frac{1}{\pi_k} F_k \left(1 + \left(\begin{matrix} \sum_U \mathbf{x}_{1k} \\ \sum_s \mathbf{x}_{2k}/\pi_k \end{matrix} - \sum_r \frac{F_k \mathbf{x}_k}{\pi_k} \right)' \left(\sum_r \frac{F_k \mathbf{x}_k \mathbf{x}'_k}{\pi_k} \right)^{-1} c_k \mathbf{x}_k \right)$$

Two-step calibration, bottom up means that from starting weights $\frac{1}{\pi_k} F_k$*, temporary weights* v_{0k} *are calculated so that the condition* $\sum_r v_{0k} \mathbf{x}_{2k} = \sum_s \frac{\mathbf{x}_{2k}}{\pi_k}$ *holds. Based on these, final weights* v_k *are calculated under the condition* $\sum_r v_k \mathbf{x}_{1k} = \sum_U \mathbf{x}_{1k}$*. The final weights are*

$$v_k = v_{0k} \left(1 + \left(\sum_U \mathbf{x}_{1k} - \sum_r v_{0k} \mathbf{x}_{1k} \right)' \left(\sum_r v_{0k} \mathbf{x}_{1k} \mathbf{x}'_{1k} \right)^{-1} c_k \mathbf{x}_{1k} \right)$$

Only the second calibration equation will be fulfilled.

In two-step calibration, top down the starting weights are $1/\pi_k$. *Temporary weights* v_{1k} *are calculated under the condition that* $\sum_s v_{1k}\mathbf{x}_{1k} = \sum_U \mathbf{x}_{1k}$. *Based on these, final weights* v_k *are calculated under the condition that* $\sum_r v_k\mathbf{x}_k = \sum_s v_{1k}\mathbf{x}_k$. *The final weights are*

$$v_k = v_{1k}\left(1 + \left(\sum_s v_{1k}\mathbf{x}_k - \sum_r v_{1k}\mathbf{x}_k\right)' \left(\sum_s v_{1k}\mathbf{x}_k\mathbf{x}_k'\right)^{-1}\mathbf{x}_k\right)$$

with v_{1k} *given by* (6).

4 Evaluating the nonresponse bias at time a

4.1 The approach

General expressions for the nonresponse bias are derived under the assumed "true" response distribution described in section 2.1. Let $R_{k|s}^{(a)}$ be a response indicator variable that, given s, takes the value 1 if $k \in r^{(a)}$, and the value 0 otherwise. We then have $E_{RD^{(a)}}\left(R_{k|s}^{(a)}\right) = \theta_{k|s}^{(a)}$, where $E_{RD^{(a)}}$ denotes expectation with respect to the unknown response distribution at time a.

The bias is expressed as an expected value over design and response distribution of the difference between the nonresponse estimator and the corresponding full response estimator. Due to the nature of the response homogeneity groups, the expectation over design is not evaluated.

Let $\hat{t}_y^{(a)}$ be an estimator for the population total t_y, based on response set $r^{(a)}$, and let $\hat{t}_{ys}$ be the corresponding (approximately) unbiased full response estimator. The bias for $\hat{t}_y^{(a)}$ is then expressed as

$$B(\hat{t}_y^{(a)}) = E_p\left[E_{RD^{(a)}}(\hat{t}_y^{(a)})\right] - t_y = E_p\left[E_{RD^{(a)}}(\hat{t}_y^{(a)}) - \hat{t}_{ys}\right] \tag{8}$$

The bias expressions derived below form the starting point for the study of change in bias between two different points in time, given in section 5.

4.2 The simple RHG estimator

A full response estimator corresponding to the simple RHG estimator $\hat{t}_{yc\pi^*}^{(a)}$ is the Horvitz-Thompson estimator

$$\hat{t}_{y\pi} = \sum_s \check{y}_k = \sum_{h=1}^{H_s} \sum_{s_h} \check{y}_k \tag{9}$$

The bias of $\hat{t}^{(a)}_{yc\pi*}$ when the response homogeneity groups model does not coincide with the true response distribution, is then, starting from (8), given by

$$B(\hat{t}^{(a)}_{yc\pi^*}) \doteq E_p\left[\sum_{h=1}^{H_s}\left(\frac{n_h}{\sum_{s_h}\theta^{(a)}_{k|s}}\sum\nolimits_{s_h}\theta^{(a)}_{k|s}\check{y}_k - \sum\nolimits_{s_h}\check{y}_k\right)\right]$$
$$= E_p\left[\sum_{h=1}^{H_s}(n_h-1)\frac{S_{\theta^{(a)}\check{y}s_h}}{\bar{\theta}^{(a)}_{s_h}}\right] \tag{10}$$

where $S_{\theta^{(a)}\check{y}s_h} = \dfrac{1}{n_h-1}\left(\sum_{s_h}\theta^{(a)}_{k|s}\check{y}_k - \dfrac{1}{n_h}\left(\sum_{s_h}\theta^{(a)}_{k|s}\right)\left(\sum_{s_h}\check{y}_k\right)\right)$ and $\bar{\theta}^{(a)}_{s_h} = \dfrac{1}{n_h}\sum_{s_h}\theta^{(a)}_{k|s}$. The bias expression (10) is approximate, valid for large response homogeneity groups. The derivation is given in appendix A.

It is easily seen that the approximate bias is zero when the RHG model coincides with the true response distribution. For a given covariance within groups, the bias is smaller for higher average response probabilities within groups. It is not certain that the approximate bias will decrease with higher average response probabilities within groups since the effect on the bias will also depend on how the covariance changes. The approximate bias will be zero (small) if $\check{y}_k$ is (approximately) constant within groups (this can be aimed at with e.g. a πps-design or stratified sampling).

4.3 The regression based RHG estimators

General case

With full response we no longer have a two-phase selection, and thus the predictions $\hat{y}_k$ cancel out, leaving

$$\hat{t}_{yreg} = \sum\nolimits_U \hat{y}_{1k} + \sum\nolimits_s \frac{y_k - \hat{y}_{1k}}{\pi_k} \tag{11}$$

$$= \sum\nolimits_U \mathbf{x}'_{1k}\hat{\mathbf{B}}_{1s} + \sum\nolimits_s \frac{y_k - \mathbf{x}'_{1k}\hat{\mathbf{B}}_{1s}}{\pi_k} \tag{12}$$

with

$$\hat{\mathbf{B}}_{1s} = \left(\sum\nolimits_s \frac{\mathbf{x}_{1k}\mathbf{x}'_{1k}}{\sigma^2_{1k}\pi_k}\right)^{-1}\sum\nolimits_s \frac{\mathbf{x}_{1k}y_k}{\sigma^2_{1k}\pi_k} \tag{13}$$

as the full response estimator corresponding to $\hat{t}^{(a)}_{ycreg}$. We then have the nonresponse bias

$$B(\hat{t}^{(a)}_{ycreg}) \doteq E_p\left[\sum_{h=1}^{H_s}(n_h-1)\frac{1}{\bar{\theta}^{(a)}_{s_h}}\left(S_{\theta\check{e}_s s_h} - S_{\theta^{(a)}\check{\mathbf{x}} s_h}\left(\hat{\mathbf{B}}^{(a)}_{\theta} - \hat{\mathbf{B}}_s\right)\right)\right] \tag{14}$$

where $\hat{\mathbf{B}}_s = \left(\sum_s \frac{\mathbf{x}_k\mathbf{x}'_k}{\sigma^2_k\pi_k}\right)^{-1}\sum_s \frac{\mathbf{x}_k y_k}{\sigma^2_k\pi_k}$, the full response analogue to $\hat{\mathbf{B}}^{(a)}_r$, $S_{\theta^{(a)}\check{e}_s s_h} = \frac{1}{n_h-1}\left(\sum_{s_h}\theta^{(a)}_{k|s}\check{e}_{ks} - \frac{1}{n_h}\left(\sum_{s_h}\theta^{(a)}_{k|s}\right)\left(\sum_{s_h}\check{e}_{ks}\right)\right)$
with $e_{ks} = y_k - \mathbf{x}'_k\hat{\mathbf{B}}_s$, $S_{\theta^{(a)}\check{\mathbf{x}} s_h}$ is analogous to $S_{\theta^{(a)}\check{e}_s s_h}$ and

$$\begin{aligned}\hat{\mathbf{B}}^{(a)}_{\theta} &= \left(\sum_{h=1}^{H_s}\frac{1}{\bar{\theta}^{(a)}_{s_h}}\sum_{s_h}\frac{\theta^{(a)}_{k|s}\mathbf{x}_k\mathbf{x}'_k}{\sigma^2_k\pi_k}\right)^{-1}\left(\sum_{h=1}^{H_s}\frac{1}{\bar{\theta}^{(a)}_{s_h}}\sum_{s_h}\frac{\theta^{(a)}_{k|s}\mathbf{x}_k y_k}{\sigma^2_k\pi_k}\right) \\ &\doteq E_{RD^{(a)}}\left(\hat{\mathbf{B}}^{(a)}_r\right)\end{aligned} \tag{15}$$

Detailed derivations are given i appendix A. It should be noted that $\mathbf{x}_{1k}$ as used in the predictions $\hat{y}^{(a)}_{1k}$ has no effect on the approximate nonresponse bias. The reason is that $\hat{y}_{1k}$ enters the formula for $\hat{t}^{(a)}_{ycreg}$ only in the terms $\sum_U \hat{y}^{(a)}_{1k} - \sum_s \hat{y}^{(a)}_{1k}/\pi_k$, with expected value approximately zero. Expression (14) for the nonresponse bias is approximate, valid for large response homogeneity groups. This is shown, using Taylor linearization, in appendix B.

We have that

$$\hat{\mathbf{B}}^{(a)}_{\theta} - \hat{\mathbf{B}}_s = \left(\sum_{h=1}^{H_s}\frac{1}{\bar{\theta}^{(a)}_{s_h}}\sum_{s_h}\frac{\theta^{(a)}_{k|s}\mathbf{x}_k\mathbf{x}'_k}{\sigma^2_k\pi_k}\right)^{-1}\left(\sum_{h=1}^{H_s}\frac{1}{\bar{\theta}^{(a)}_{s_h}}\sum_{s_h}\frac{\theta^{(a)}_{k|s}\mathbf{x}_k e_{ks}}{\sigma^2_k\pi_k}\right) \tag{16}$$

From this and expression (14), it is easily seen that the nonresponse bias will be approximately zero if the residuals $e_{ks} = y_k - \mathbf{x}'_k\hat{\mathbf{B}}_s = 0$ for all k, since in that case both $S_{\theta^{(a)}\check{e}_s s_h} = 0$ and $\hat{\mathbf{B}}^{(a)}_{\theta} - \hat{\mathbf{B}}_s$ are zero, even if the RHG model is not true. The bias will also be approximately zero (small) if $\check{e}_{ks}$ is (approximately) constant within groups.

Special case 1

In this case the only auxiliary information is $\mathbf{x}_{1k}$, with $\sum_U \mathbf{x}_{1k}$ known. The full response estimator corresponding to $\hat{t}^{(a)}_{ycreg1}$ is the same as in the general case. For large response homogeneity groups, the nonresponse bias for $\hat{t}^{(a)}_{ycreg1}$ is then given approximately by (a derivation is given in appendix A) the same bias expression as in the general case, except that a different auxiliary vector, $\mathbf{x}_1$ instead of $\mathbf{x}$, is used. Thus the same conclusions on zero approximate bias applies.

Special case 2

The auxiliary information in this case is $\mathbf{x}_k$, available only for $k \in s$. In the situation with full response, this means that no auxiliary information can be used to form a regression estimator. Consequently, a corresponding full response estimator is the HT-estimator, given by (9). The resulting approximate expression for the nonresponse bias is the same as in the general case. At first glance this may seem unreasonable since $\hat{t}^{(a)}_{ycreg}$ uses additional auxiliary information, but the reason for this should be clear from the derivation of the nonresponse bias in the general case. The sampling variance will be different, however.

4.4 The calibration estimator

Special case 1

The auxiliary information to be used is $\mathbf{x}_{1k}$. The calibration estimator given by setting $v^{(a)}_{kr} = v^{(a)}_{1kr}$ in (5) can be rewritten as

$$\hat{t}^{(a)}_{ycal1} = \textstyle\sum_U \mathbf{x}'_{1k}\tilde{\mathbf{B}}^{(a)}_{1r} + \textstyle\sum_{r^{(a)}} \frac{y_k - \mathbf{x}'_{1k}\tilde{\mathbf{B}}^{(a)}_{1r}}{\pi_k} \tag{17}$$

with

$$\tilde{\mathbf{B}}^{(a)}_{1r} = \left(\textstyle\sum_{r^{(a)}} \frac{c_k \mathbf{x}_{1k}\mathbf{x}'_{1k}}{\pi_k}\right)^{-1} \left(\textstyle\sum_{r^{(a)}} \frac{c_k \mathbf{x}_{1k} y_k}{\pi_k}\right) \tag{18}$$

The corresponding full response estimator is the same as for the regression based RHG estimator in case 1, since (17) and (3) are equivalent if there is full response and we choose $c_k = 1/\sigma^2_{1k}$. After some algebra, the approximate bias of $\hat{t}^{(a)}_{ycal1}$ can be expressed as

$$\begin{aligned} B(\hat{t}^{(a)}_{ycal1}) &\doteq E_p\left[\textstyle\sum_s \left(1-\theta^{(a)}_{k|s}\right)\frac{\mathbf{x}'_{1k}}{\pi_k}\left(\tilde{\mathbf{B}}^{(a)}_{1\theta} - \hat{\mathbf{B}}_{1s}\right) - \textstyle\sum_s (1-\theta^{(a)}_{k|s})\frac{e_{1ks}}{\pi_k}\right] \\ &= E_p\left[\textstyle\sum_s \left(1-\theta^{(a)}_{k|s}\right)\left(\frac{\mathbf{x}'_{1k}}{\pi_k}\left(\tilde{\mathbf{B}}^{(a)}_{1\theta} - \hat{\mathbf{B}}_{1s}\right) - \frac{e_{1ks}}{\pi_k}\right)\right] \end{aligned} \tag{19}$$

where $\tilde{\mathbf{B}}_{1\theta}^{(a)} = \left(\sum_s \frac{c_k \theta_{k|s}^{(a)} \mathbf{x}_{1k} \mathbf{x}_{1k}'}{\pi_k}\right)^{-1} \left(\sum_s \frac{c_k \theta_{k|s}^{(a)} \mathbf{x}_{1k} y_k}{\pi_k}\right) \doteq E_p\left(\tilde{\mathbf{B}}_{1r}^{(a)}\right)$ and $e_{1ks} = y_k - \mathbf{x}_{1k}'\hat{\mathbf{B}}_{1s}$. The derivation of (19) is shown in appendix A.

Remark 4 *Rewriting the bias expression (19) gives the alternative bias expression*

$$E_p\left[-\sum_s \left(1 - \theta_{k|s}^{(a)}\right) \frac{y_k - \mathbf{x}_{1k}'\tilde{\mathbf{B}}_{1\theta}^{(a)}}{\pi_k}\right] = E_p\left[-\sum_s \left(1 - \theta_{k|s}^{(a)}\right) \frac{e_{1k\theta}^{(a)}}{\pi_k}\right] \tag{20}$$

where $e_{1k\theta}^{(a)} = y_k - \mathbf{x}_{1k}'\tilde{\mathbf{B}}_{1\theta}^{(a)}$. *If the response distribution is such that the response probabilities are independent of the realized sample and are defined for every* $k \in U$, *i.e. the response probabilities are* $\theta_k^{(a)}$, *we can evaluate the expectation with respect to the sampling distribution. The resulting bias expression is given in Lundström (1997). That situation could perhaps be realistic in a postal survey if the entire follow-up procedure is determined before the sample is selected.*

We have that

$$\tilde{\mathbf{B}}_{1\theta}^{(a)} - \hat{\mathbf{B}}_{1s} = \left(\sum_s \frac{c_k \theta_{k|s}^{(a)} \mathbf{x}_{1k} \mathbf{x}_{1k}'}{\pi_k}\right)^{-1} \left(\sum_s \frac{c_k \theta_{k|s}^{(a)} \mathbf{x}_{1k} e_{1ks}}{\pi_k}\right) \tag{21}$$

Since $\tilde{\mathbf{B}}_{1\theta}^{(a)} - \hat{\mathbf{B}}_{1s} = 0$ if $e_{1ks} = 0$ for every k, it is obvious that the approximate nonresponse bias will be zero in that case. The approximate bias will also be zero if $\theta_{k|s}^{(a)} = 1 + c_k \boldsymbol{\lambda}' \mathbf{x}_{1k}$ for some constant vector $\boldsymbol{\lambda}$. This would be the case if, for example, $\mathbf{x}_{1k}$ is a group identifier and the response probabilities are constant within those groups. The calibration estimator with such an auxiliary vector is similar to the simple RHG estimator. A more detailed discussion of the approximate bias in the special case of Remark 4 is given in Lundström (1997).

Special case 2

In this case, the calibration estimator can be rewritten as

$$\hat{t}_{ycal2}^{(a)} = \sum_s \frac{\mathbf{x}_k'\tilde{\mathbf{B}}_r^{(a)}}{\pi_k} + \sum_{r^{(a)}} \frac{y_k - \mathbf{x}_k'\tilde{\mathbf{B}}_r^{(a)}}{\pi_k} \tag{22}$$

with

$$\tilde{\mathbf{B}}_r^{(a)} = \left(\sum\nolimits_{r^{(a)}} \frac{c_k \mathbf{x}_k \mathbf{x}_k'}{\pi_k} \right)^{-1} \left(\sum\nolimits_{r^{(a)}} \frac{c_k \mathbf{x}_k y_k}{\pi_k} \right) \tag{23}$$

Since there is no auxiliary information above the sample level, the full response estimator is the HT-estimator $\hat{t}_{y\pi}$. The resulting nonresponse bias is the same as in special case 1, but based on $\mathbf{x}$ instead of $\mathbf{x}_1$. A proof is given in appendix A. The conclusions on zero approximate bias in case 1 also apply here.

5 Change in nonresponse bias between two points of time

From the approximate expressions for the nonresponse bias in section 4, we see that the true response probabilities are instrumental in determining the estimator bias. To conclude what happens to the bias as the data collection proceeds, we must compare the biases for estimators from different stages in the data collection process. By studying the difference in approximate bias between two points of time, conclusions can be drawn about under what conditions it remains unchanged between those two occasions, given our assumptions about the true response mechanism.

The difference in approximate bias between estimators $\hat{t}_y^{(a-1)}$ and $\hat{t}_y^{(a)}$ is

$$B(\hat{t}_y^{(a)}) - B(\hat{t}_y^{(a-1)}) = E_p \left[E_{RD^{(a)}}(\hat{t}_y^{(a)}) - E_{RD^{(a-1)}}(\hat{t}_y^{(a-1)}) \middle| \, s \right]$$

We give expressions for the bias difference for each of the studied estimators and conclude with some simple examples to show how the bias is influenced by changes in the response probabilities.

5.1 The simple RHG estimator

For the simple RHG estimator it is easy to show that the bias difference is

$$B(\hat{t}^{(a)}_{yc\pi^*}) - B(\hat{t}^{(a-1)}_{yc\pi^*}) \doteq E_p\left[\sum_{h=1}^{H_s}(n_h-1)\frac{S_{\theta^{(a)}\check{y}s_h}}{\bar{\theta}^{(a)}_{s_h}} - \sum_{h=1}^{H_s}(n_h-1)\frac{S_{\theta^{(a-1)}\check{y}s_h}}{\bar{\theta}^{(a-1)}_{s_h}}\right] \tag{24}$$

$$= E_p\left[\sum_{h=1}^{H_s}(n_h-1)S_{\check{y}s_h}\left(\frac{r_{\theta^{(a)}\check{y}s_h}S_{\theta^{(a)}s_h}}{\bar{\theta}^{(a)}_{s_h}} - \frac{r_{\theta^{(a-1)}\check{y}s_h}S_{\theta^{(a-1)}s_h}}{\bar{\theta}^{(a-1)}_{s_h}}\right)\right] \tag{25}$$

where $r_{\theta^{(a)}\check{y}s_h} = S_{\theta^{(a)}\check{y}s_h}/(S_{\theta^{(a)}s_h}S_{\check{y}s_h})$, with $S_{\theta^{(a)}s_h}$ and $S_{\check{y}s_h}$ being the standard deviations of $\theta^{(a)}$ and $\check{y}$ in s_h.

It is not obvious in what cases the difference will be zero, negative or positive. Whether the nonresponse biases at time a and at time $a-1$ will differ depends on how $r_{\theta\check{y}s_h}$ and S_θ change. As an illustration, some simple examples are given in section 5.4

5.2 The regression based RHG estimators

General case

From the approximate expression for the nonresponse bias we see that the difference in bias between time $a-1$ and a can be expressed as

$$\begin{aligned}
B(\hat{t}^{(a)}_{ycreg}) - B(\hat{t}^{(a-1)}_{ycreg}) &\doteq E_p\Bigg[\sum_{h=1}^{H_s}(n_h-1)\frac{1}{\bar{\theta}^{(a)}_{s_h}}\left(S_{\theta^{(a)}\check{e}_s s_h} - S_{\theta^{(a)}\check{\mathbf{x}}s_h}\right)\left(\hat{\mathbf{B}}^{(a)}_\theta - \hat{\mathbf{B}}_s\right) \\
&\quad - \sum_{h=1}^{H_s}(n_h-1)\frac{1}{\bar{\theta}^{(a-1)}_{s_h}}\left(S_{\theta^{(a-1)}\check{e}_s s_h} - S_{\theta^{(a-1)}\check{\mathbf{x}}s_h}\right)\left(\hat{\mathbf{B}}^{(a-1)}_\theta - \hat{\mathbf{B}}_s\right)\Bigg] \\
&= E_p\Bigg[\sum_{h=1}^{H_s}(n_h-1)\Bigg(S_{\check{e}_s s_h}\left\{\frac{r_{\theta^{(a)}\check{e}_s s_h}S_{\theta^{(a)}s_h}}{\bar{\theta}^{(a)}_{s_h}} - \frac{r_{\theta^{(a-1)}\check{e}_s s_h}S_{\theta^{(a-1)}s_h}}{\bar{\theta}^{(a-1)}_{s_h}}\right\} \\
&\quad - S_{\check{\mathbf{x}}s_h}\left\{\frac{r_{\theta^{(a)}\check{\mathbf{x}}s_h}S_{\theta^{(a)}s_h}\left(\hat{\mathbf{B}}^{(a)}_\theta - \hat{\mathbf{B}}_s\right)}{\bar{\theta}^{(a)}_{s_h}} - \frac{r_{\theta^{(a-1)}\check{\mathbf{x}}s_h}S_{\theta^{(a-1)}s_h}\left(\hat{\mathbf{B}}^{(a)}_\theta - \hat{\mathbf{B}}_s\right)}{\bar{\theta}^{(a-1)}_{s_h}}\right\}\Bigg)\Bigg]
\end{aligned} \tag{26}$$

where $r_{\theta^{(a)}\check{e}_s s_h}$ and $r_{\theta^{(a)}\check{\mathbf{x}} s_h}$ are sample correlation coefficients.

In this case it is even more difficult to draw conclusions about how the approximate bias changes (or when it does not) as a result of changes in the response probabilities. Simple examples are given in section 5.4.

Special cases

In special case 2, the approximate nonresponse bias is the same as in the general case, so the approximate bias difference is given by (26). This is also the approximate bias difference in special case 1, but with $\mathbf{x}_{1k}$ instead of $\mathbf{x}_k$.

5.3 The calibration estimators

Special case 1

From the approximate bias expression (19), we get the bias difference

$$B(\hat{t}^{(a)}_{ycal1}) - B(\hat{t}^{(a-1)}_{ycal1}) = E_p\left[\sum\nolimits_s(\theta^{(a)}_{k|s} - \theta^{(a-1)}_{k|s})\frac{e_{1ks}}{\pi_k} + \sum\nolimits_s \frac{\mathbf{x}'_{1k}}{\pi_k}\left(\tilde{\mathbf{B}}^{(a)}_{1\theta} - \tilde{\mathbf{B}}^{(a-1)}_{1\theta}\right)\right.$$
$$\left. -\sum\nolimits_s\left(\frac{\theta^{(a)}_{k|s}\mathbf{x}'_{1k}}{\pi_k}\left(\tilde{\mathbf{B}}^{(a)}_{1\theta} - \hat{\mathbf{B}}_{1s}\right) - \frac{\theta^{(a-1)}_{k|s}\mathbf{x}'_{1k}}{\pi_k}\left(\tilde{\mathbf{B}}^{(a-1)}_{1\theta} - \hat{\mathbf{B}}_{1s}\right)\right)\right] \quad (27)$$

The interpretation of (27) is far from trivial, but we can see that if both $\tilde{\mathbf{B}}^{(a-1)}_{1\theta}$ and $\tilde{\mathbf{B}}^{(a)}_{1\theta}$ are equal to $\hat{\mathbf{B}}_{1s}$ (correspondingly in case 2) at both time $a-1$ and a, then the last two terms cancel out, leaving the difference as a product of the increase in response probabilities and the residuals.

Special case 2

Since the approximate nonresponse bias is the same as in case 1, the approximate bias difference is given by (27).

5.4 Some simple examples

To see how the approximate bias difference is influenced by changes in the response probabilities, consider the following simple (somewhat unrealistic) examples.

Example 1 Assume that $\theta_{k|s}^{(a)} = \theta_{k|s}^{(a-1)} + \alpha_h$, i.e. that the response probabilities increase with the same amount for all elements in the same group between time $a-1$ and a. This means that $r_{\theta^{(a)}\check{y}s_h} = r_{\theta^{(a-1)}\check{y}s_h}$, $S_{\theta^{(a)}s_h} = S_{\theta^{(a-1)}s_h}$ and $\bar{\theta}_{s_h}^{(a)} > \bar{\theta}_{s_h}^{(a-1)}$. For the simple RHG estimator, this leads to a negative approximate bias difference. We can not automatically draw the conclusion that the approximate nonresponse bias is smaller at time a, since one or both of the biases may be negative.

For the regression based RHG estimators, we see that $r_{\theta^{(a)}\check{e}_s s_h} = r_{\theta^{(a-1)}\check{e}_s s_h}$. Inserting this into the expression (26) for the bias difference in the general case, and rearranging terms, we see that the difference will be

$$E_p\left[(n_h-1)S_{\theta^{(a-1)}s_h}\left\{S_{\check{e}_s s_h} r_{\theta^{(a-1)}\check{e}_s s_h}\left(\frac{1}{\bar{\theta}_{s_h}^{(a)}} - \frac{1}{\bar{\theta}_{s_h}^{(a-1)}}\right)\right.\right.$$
$$\left.\left. - S_{\check{\mathbf{x}}s_h} r_{\theta^{(a-1)}\check{\mathbf{x}}s_h}\left(\frac{\hat{\mathbf{B}}_{\theta e_s}^{(a)}}{\bar{\theta}_{s_h}^{(a)}} - \frac{\hat{\mathbf{B}}_{\theta e_s}^{(a-1)}}{\bar{\theta}_{s_h}^{(a-1)}}\right)\right\}\right] \tag{28}$$

Since the approximate bias and bias difference for the calibration estimator depend on the formulation of the auxiliary vector, it is not illustrative to apply the example directly. However, if $\mathbf{x}_1$ ($\mathbf{x}$ in case 2) is a group identifier with non-overlapping groups, they will play a role similar to that of the response homogeneity groups.

Example 2 Assume that $\theta_{k|s}^{(a)} = (1+\delta_h)\theta_{k|s}^{(a-1)}$, i.e. that the response probabilities increase by the same factor for all elements in the same group. We then have $r_{\theta^{(a)}\check{y}s_h} = r_{\theta^{(a-1)}\check{y}s_h}$, $S_{\theta^{(a)}s_h} = (1+\delta_h)S_{\theta^{(a-1)}s_h}$, $\bar{\theta}_{s_h}^{(a)} = (1+\delta_h)\bar{\theta}_{s_h}^{(a-1)}$ and $\hat{\mathbf{B}}_{\theta e_s}^{(a)} = \hat{\mathbf{B}}_{\theta e_s}^{(a-1)}$. It is easily seen that this leads to an approximate bias difference of 0, i.e. no reduction of the approximate bias between time $a-1$ and a, for any of the RHG estimators.

Example 3 Assume that $\theta_{k|s}^{(a)} = (1+\delta)\theta_{k|s}^{(a-1)}$. This means that $\tilde{\mathbf{B}}_{1\theta}^{(a)} = \tilde{\mathbf{B}}_{1\theta}^{(a-1)}$, which gives the approximate bias difference

$$B(\hat{t}_{ycal1}^{(a)}) - B(\hat{t}_{ycal1}^{(a-1)}) = \textstyle\sum_s \delta\theta_{k|s}^{(a-1)}\frac{e_{1ks}}{\pi_k} - \sum_s \delta\theta_{k|s}^{(a-1)}\frac{\mathbf{x}'_{1ks}}{\pi_k}\left(\tilde{\mathbf{B}}_{1\theta}^{(a-1)} - \hat{\mathbf{B}}_{1s}\right)$$

6 Conclusions and future work

Under the assumed true response distribution, it is not obvious that an increase in response probabilities leads to a reduction of the approximate nonresponse bias for any of the studied estimators.

With strong auxiliary information, the nonresponse bias can be reduced, but this will generally mean that the difference in bias will be smaller, i.e. the effect of reductive efforts are smaller.

Obviously, if the approximate nonresponse bias is zero or negligible (at any given time) then the change in bias will also be zero or negligible. This means that we gain nothing in terms of bias by trying to increase the response probabilities, so we can focus on precision (estimator variances).

As stated in the introduction, the purpose of this research is to consider the balance between the cost and the effect of nonresponse reduction, a purpose that can only be met by joint cost and error modeling. This paper deals with one aspect of the error, namely the effect of nonresponse rate reduction efforts on the nonresponse bias. In the presence of nonresponse, bias is the most important factor, variance reduction becomes a secondary goal. In a situation where we consider leaving out nonresponse reducing efforts, variance is however an important issue, since it can become prohibitively large if the nonresponse is allowed to increase. We will also need to find point and variance estimators for the bias (and variance) differences.

So far, only a single study variable and the estimation of a total have been considered. In most surveys there are several study variables and many more parameters to be estimated, including functions of totals, issues that involve considerations that need to be studied further.

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A Nonresponse bias at time a

The nonresponse biases are derived under the assumptions about the response distribution given in section 2.1.

A.1 The simple RHG estimator

The nonresponse bias for $\hat{t}^{(a)}_{yc\pi^*}$ is

$$\begin{aligned}
B(\hat{t}^{(a)}_{yc\pi^*}) &= E_p E_{RD_a}\left[\hat{t}^{(a)}_{yc\pi^*}\right] - t_y \\
&= E_p\left[E_{RD_a}(\hat{t}^{(a)}_{yc\pi^*}) - \hat{t}_{y\pi}\right] \\
&= E_p\left[E_{RD^{(a)}}\left(\sum_{h=1}^{H_s}\frac{n_h}{m_h^{(a)}}\sum\nolimits_{r_h}\check{y}_k\right) - \sum_{h=1}^{H_s}\sum\nolimits_{s_h}\check{y}_k\right]
\end{aligned}$$

By using $R^{(a)}_{k|s}$, the response indicator variables, we can express $m_h^{(a)}$ as $\sum_{s_h} R^{(a)}_{k|s}$ and $\sum_{r_h}\check{y}_k$ as $\sum_{s_h} R^{(a)}_{k|s}\check{y}_k$, which gives

$$B(\hat{t}^{(a)}_{yc\pi^*}) = E_p\left[E_{RD_a}\left(\sum_{h=1}^{H_s}\frac{n_h}{\sum_{s_h} R^{(a)}_{k|s}}\sum\nolimits_{s_h} R^{(a)}_{k|s}\check{y}_k\right) - \sum_{h=1}^{H_s}\sum\nolimits_{s_h}\check{y}_k\right] \tag{A.1}$$

$$\begin{aligned}
&\doteq E_p\left[\sum_{h=1}^{H_s}\left(\frac{n_h}{\sum_{s_h}\theta^{(a)}_{k|s}}\sum\nolimits_{s_h}\theta^{(a)}_{k|s}\check{y}_k - \sum\nolimits_{s_h}\check{y}_k\right)\right] \\
&= E_p\left[\sum_{h=1}^{H_s}\frac{n_h}{\sum_{s_h}\theta^{(a)}_{k|s}}\left(\sum\nolimits_{s_h}\theta^{(a)}_{k|s}\check{y}_k - \frac{1}{n_h}\left(\sum\nolimits_{s_h}\theta^{(a)}_{k|s}\right)\left(\sum\nolimits_{s_h}\check{y}_k\right)\right)\right] \\
&= E_p\left[\sum_{h=1}^{H_s}(n_h - 1)\frac{S_{\theta^{(a)}\check{y}s_h}}{\bar{\theta}^{(a)}_{s_h}}\right]
\end{aligned}$$

Since we make use of the large-sample approximation that the expected value of a ratio of random variables equals the ratio of the corresponding expected values, the bias expression is approximate, valid for large response homogeneity groups.

A.2 The regression based RHG estimators

A.2.1 General case

The nonresponse bias for $\hat{t}^{(a)}_{ycreg}$ is

$$
\begin{aligned}
B(\hat{t}^{(a)}_{ycreg}) &= E_p E_{RD^{(a)}}\left[\hat{t}^{(a)}_{ycreg}\right] - t_y \doteq E_p\left[E_{RD^{(a)}}\left(\hat{t}^{(a)}_{ycreg}\right) - \hat{t}_{yreg}\right] \\
&= E_p\Bigg[E_{RD^{(a)}}\Bigg(\textstyle\sum_U \mathbf{x}'_{1k}\hat{\mathbf{B}}^{(a)}_{1r} + \sum_s \dfrac{\mathbf{x}'_k\hat{\mathbf{B}}^{(a)}_r - \mathbf{x}'_{1k}\hat{\mathbf{B}}^{(a)}_{1r}}{\pi_k} \\
&\quad + \sum_{h=1}^{H_s}\frac{n_h}{\sum_{s_h} R^{(a)}_{k|s}}\textstyle\sum_{s_h}\dfrac{R^{(a)}_{k|s}\left(y_k - \mathbf{x}'_k\hat{\mathbf{B}}^{(a)}_r\right)}{\pi_k}\Bigg) - \textstyle\sum_U \mathbf{x}'_{1k}\hat{\mathbf{B}}_{1s} - \sum_s \dfrac{y_k - \mathbf{x}'_{1k}\hat{\mathbf{B}}_{1s}}{\pi_k}\Bigg] \\
&= E_p\Bigg[\textstyle\sum_U \mathbf{x}'_{1k}\left(E_{RD^{(a)}}\left(\hat{\mathbf{B}}^{(a)}_{1r}\right) - \hat{\mathbf{B}}_{1s}\right) - \sum_s \dfrac{\mathbf{x}'_{1k}}{\pi_k}\left(E_{RD^{(a)}}\left(\hat{\mathbf{B}}^{(a)}_{1r}\right) - \hat{\mathbf{B}}_{1s}\right) \\
&\quad - E_{RD^{(a)}}\left(\textstyle\sum_s \dfrac{y_k - \mathbf{x}'_k\hat{\mathbf{B}}^{(a)}_r}{\pi_k} - \displaystyle\sum_{h=1}^{H_s}\frac{n_h}{\sum_{s_h} R^{(a)}_{k|s}}\textstyle\sum_{s_h}\dfrac{R^{(a)}_{k|s}\left(y_k - \mathbf{x}'_k\hat{\mathbf{B}}^{(a)}_r\right)}{\pi_k}\right)\Bigg]
\end{aligned}
\tag{A.2}
$$

Now, we have that

$$
E_p\left[\textstyle\sum_U \mathbf{x}'_{1k}\left(E_{RD^{(a)}}\left(\hat{\mathbf{B}}^{(a)}_{1r}\right) - \hat{\mathbf{B}}_{1s}\right)\right] \doteq E_p\left[\textstyle\sum_s \dfrac{\mathbf{x}'_{1k}}{\pi_k}\left(E_{RD^{(a)}}\left(\hat{\mathbf{B}}^{(a)}_{1r}\right) - \hat{\mathbf{B}}_{1s}\right)\right]
\tag{A.3}
$$

so the first two terms of (A.2) cancel out, leaving

$$
B(\hat{t}^{(a)}_{ycreg}) \doteq E_p E_{RD^{(a)}}\left(\sum_{h=1}^{H_s}\frac{n_h}{\sum_{s_h} R^{(a)}_{k|s}}\textstyle\sum_{s_h}\dfrac{R^{(a)}_{k|s}\left(y_k - \mathbf{x}'_k\hat{\mathbf{B}}^{(a)}_r\right)}{\pi_k} - \textstyle\sum_s \dfrac{y_k - \mathbf{x}'_k\hat{\mathbf{B}}^{(a)}_r}{\pi_k}\right)
\tag{A.4}
$$

By adding and subtracting $\sum_s \frac{\mathbf{x}_k' \hat{\mathbf{B}}_s}{\pi_k}$, we get

$$\begin{aligned}
B(\hat{t}_{ycreg}^{(a)}) \doteq & E_p \left[\sum_s \frac{\mathbf{x}_k'}{\pi_k} \left(E_{RD^{(a)}} \left(\hat{\mathbf{B}}_r^{(a)} \right) - \hat{\mathbf{B}}_s \right) - \sum_s \frac{y_k - \mathbf{x}_k' \hat{\mathbf{B}}_s}{\pi_k} \right. \\
& \left. + E_{RD^{(a)}} \left(\sum_{h=1}^{H_s} \frac{n_h}{\sum_{s_h} R_{k|s}^{(a)}} \sum_{s_h} \frac{R_{k|s}^{(a)} \left(y_k - \mathbf{x}_k' \hat{\mathbf{B}}_r^{(a)} \right)}{\pi_k} \right) \right] \qquad \text{(A.5)} \\
\doteq & E_p \left[\left(\sum_s \frac{\mathbf{x}_k'}{\pi_k} - \sum_{h=1}^{H_s} \frac{1}{\bar{\theta}_{s_h}^{(a)}} \sum_{s_h} \frac{\theta_{k|s}^{(a)} \mathbf{x}_k'}{\pi_k} \right) \left(\hat{\mathbf{B}}_\theta^{(a)} - \hat{\mathbf{B}}_s \right) \right. \\
& \left. + \sum_{h=1}^{H_s} \left(\frac{1}{\bar{\theta}_{s_h}^{(a)}} \sum_{s_h} \frac{\theta_{k|s}^{(a)} e_{ks}}{\pi_k} - \sum_{s_h} \frac{e_{ks}}{\pi_k} \right) \right] \qquad \text{(A.6)}
\end{aligned}$$

with $e_{ks} = y_k - \mathbf{x}_k' \hat{\mathbf{B}}_s$. Rearranging terms gives

$$\begin{aligned}
B(\hat{t}_{ycreg}^{(a)}) \doteq & E_p \left[\sum_{h=1}^{H_s} \frac{1}{\bar{\theta}_{s_h}^{(a)}} \left(\sum_{s_h} \frac{\theta_{k|s}^{(a)} e_{ks}}{\pi_k} - \frac{1}{n_h} \left(\sum_{s_h} \theta_{k|s}^{(a)} \right) \left(\sum_{s_h} \frac{e_{ks}}{\pi_k} \right) \right) \right. \\
& \left. - \sum_{h=1}^{H_s} \frac{1}{\bar{\theta}_{s_h}^{(a)}} \left(\sum_{s_h} \frac{\theta_{k|s}^{(a)} \mathbf{x}_k'}{\pi_k} - \frac{1}{n_h} \left(\sum_{s_h} \theta_{k|s}^{(a)} \right) \left(\sum_{s_h} \frac{\mathbf{x}_k'}{\pi_k} \right) \right) \left(\hat{\mathbf{B}}_\theta^{(a)} - \hat{\mathbf{B}}_s \right) \right] \\
& \qquad \text{(A.7)} \\
= & E_p \left[\sum_{h=1}^{H_s} (n_h - 1) \frac{1}{\bar{\theta}_{s_h}^{(a)}} \left(S_{\theta \check{e}_s s_h} - S_{\theta^{(a)} \check{\mathbf{x}} s_h} \left(\hat{\mathbf{B}}_\theta^{(a)} - \hat{\mathbf{B}}_s \right) \right) \right] \qquad \text{(A.8)}
\end{aligned}$$

which is expression (14).

A.2.2 Special case 1

The approximate nonresponse bias in case 1 follows easily from the general case if we note that the bias for $\hat{t}^{(a)}_{ycreg1}$ is

$$
\begin{aligned}
B(\hat{t}^{(a)}_{ycreg1}) \doteq\; & E_p\left[E_{RD^{(a)}}\left(\hat{t}^{(a)}_{ycreg1}\right) - \hat{t}_{yreg1}\right] \\
=\; & E_p\left[E_{RD^{(a)}}\left(\sum\nolimits_U \mathbf{x}'_{1k}\hat{\mathbf{B}}^{(a)}_{1r} + \sum_{h=1}^{H_s}\frac{n_h}{m^{(a)}_h}\sum\nolimits_{r^{(a)}_h}\frac{y_k - \mathbf{x}'_{1k}\hat{\mathbf{B}}^{(a)}_{1r}}{\pi_k}\right)\right. \\
& \left. -\sum\nolimits_U \mathbf{x}'_{1k}\hat{\mathbf{B}}_{1s} - \sum\nolimits_s \frac{y_k - \mathbf{x}'_{1k}\hat{\mathbf{B}}_{1s}}{\pi_k}\right]
\end{aligned}
$$

By adding and subtracting $\displaystyle\sum_{h=1}^{H_s}\frac{n_h}{m^{(a)}_h}\sum\nolimits_{s_h}\frac{R^{(a)}_{k|s}(y_k - \mathbf{x}'_{1k}\hat{\mathbf{B}}^{(a)}_{1\theta})}{\pi_k}$, we see that this is the same expression as (A.6), if we replace $\mathbf{x}_k$ with $\mathbf{x}_{1k}$ and use (A.3). Repeating the steps (A.7) and (A.8) gives the approximate bias in this case.

A.2.3 Special case 2

The approximate nonresponse bias for $\hat{t}^{(a)}_{ycreg2}$ is the same as in the general case and follows directly from (A.4), no separate proof is needed.

A.3 The calibration estimators

The derivation of the nonresponse bias for the calibration estimators follow closely those for the regression estimators.

A.3.1 Special case 1

From expression (17) for the calibration estimator, we see that the approximate nonresponse bias for $\hat{t}^{(a)}_{ycal1}$ is

$$\begin{aligned} B(\hat{t}^{(a)}_{ycal1}) \doteq & E_p\left[E_{RD^{(a)}}\left(\hat{t}^{(a)}_{ycal1}\right) - \hat{t}_{yreg}\right] \\ = & E_p\left[E_{RD^{(a)}}\left(\sum_U \mathbf{x}'_{1k}\tilde{\mathbf{B}}^{(a)}_{1r} + \sum_s \frac{R^{(a)}_{k|s}\left(y_k - \mathbf{x}'_{1k}\tilde{\mathbf{B}}^{(a)}_{1r}\right)}{\pi_k}\right)\right. \\ & \left. - \sum_U \mathbf{x}'_{1k}\hat{\mathbf{B}}_{1s} - \sum_s \frac{(y_k - \mathbf{x}'_{1k}\hat{\mathbf{B}}_{1s})}{\pi_k}\right] \end{aligned} \tag{A.9}$$

By adding and subtracting $\sum_s \dfrac{R^{(a)}_{k|s}\mathbf{x}'_{1k}\hat{\mathbf{B}}_{1s}}{\pi_k}$, we get

$$\begin{aligned} B(\hat{t}^{(a)}_{ycal1}) \doteq & E_p\left[E_{RD^{(a)}}\left(\left(\sum_U \mathbf{x}'_{1k} - \sum_s \frac{R^{(a)}_{k|s}\mathbf{x}'_{1k}}{\pi_k}\right)\left(\tilde{\mathbf{B}}^{(a)}_{1r} - \hat{\mathbf{B}}_{1s}\right)\right)\right. \\ & \left. + \sum_s \frac{R^{(a)}_{k|s}(y_k - \mathbf{x}'_{1k}\hat{\mathbf{B}}_{1s})}{\pi_k} - \sum_s \frac{(y_k - \mathbf{x}'_{1k}\hat{\mathbf{B}}_{1s})}{\pi_k}\right] \end{aligned} \tag{A.10}$$

Using that

$$E_p\left[\sum_U \mathbf{x}'_{1k}\left(E_{RD^{(a)}}\left(\tilde{\mathbf{B}}^{(a)}_{1r}\right) - \hat{\mathbf{B}}_{1s}\right)\right] \doteq E_p\left[\sum_s \frac{\mathbf{x}'_{1k}}{\pi_k}\left(E_{RD^{(a)}}\left(\tilde{\mathbf{B}}^{(a)}_{1r}\right) - \hat{\mathbf{B}}_{1s}\right)\right]$$

we get

$$\begin{aligned} B(\hat{t}^{(a)}_{ycal1}) \doteq & E_p\left[\left(\sum_s \frac{\mathbf{x}'_{1k}}{\pi_k} - \sum_s \frac{\theta^{(a)}_{k|s}\mathbf{x}'_{1k}}{\pi_k}\right)\left(\tilde{\mathbf{B}}^{(a)}_{1\theta} - \hat{\mathbf{B}}_{1s}\right) - \sum_s (1 - \theta^{(a)}_{k|s})\frac{(y_k - \mathbf{x}'_{1k}\hat{\mathbf{B}}_{1s})}{\pi_k}\right] \\ = & E_p\left[\sum_s \left(1 - \theta^{(a)}_{k|s}\right)\left(\frac{\mathbf{x}'_{1k}}{\pi_k}\left(\tilde{\mathbf{B}}^{(a)}_{1\theta} - \hat{\mathbf{B}}_{1s}\right) - \frac{e_{1ks}}{\pi_k}\right)\right] \end{aligned} \tag{A.11}$$

The expression for the approximate nonresponse bias given in Remark 4, follows easily by letting $\theta^{(a)}_{k|s} = \theta^{(a)}_k$ for $k \in U$ and evaluating the expectation.

A.3.2 Special case 2

It is easily seen that the expression for the nonresponse bias will be the same as in special case 1 if we note that the bias in this case is

$$B(\hat{t}^{(a)}_{ycal2}) = E_p\left[E_{RD^{(a)}}\left(\sum\nolimits_s \frac{\mathbf{x}'_k\tilde{\mathbf{B}}^{(a)}_r}{\pi_k} + \sum\nolimits_s \frac{R^{(a)}_{k|s}\left(y_k - \mathbf{x}'_k\tilde{\mathbf{B}}^{(a)}_r\right)}{\pi_k}\right) - \sum\nolimits_s \frac{y_k}{\pi_k}\right]$$

which, if we add and subtract $\sum_s \frac{\mathbf{x}'_k\hat{\mathbf{B}}_s}{\pi_k}$, is the same expression as (A.9), but with $\mathbf{x}$ instead of $\mathbf{x}_1$ and $\sum_s \check{\mathbf{x}}_{1k}$ instead of $\sum_U \mathbf{x}_{1k}$. Repeating the steps from (A.9) to (A.11) gives the bias in this case.

B Taylor linearizations

B.1 Nonresponse bias for the regression based RHG estimator

The nonresponse bias for the general regression based RHG estimator is given by (A.5):

$$E_pE_{RD^{(a)}}\left(\sum_{h=1}^{H_s}\frac{n_h}{\sum_{s_h}R^{(a)}_{k|s}}\sum\nolimits_{s_h}\frac{R^{(a)}_{k|s}\left(y_k - \mathbf{x}'_k\hat{\mathbf{B}}^{(a)}_r\right)}{\pi_k} - \sum\nolimits_s \frac{y_k - \mathbf{x}'_k\hat{\mathbf{B}}^{(a)}_r}{\pi_k}\right)$$

where

$$\hat{\mathbf{B}}^{(a)}_r = \left(\sum_{h=1}^{H_s}\frac{n_h}{\sum_{s_h}R^{(a)}_{k|s}}\sum\nolimits_{s_h}\frac{R^{(a)}_{k|s}\mathbf{x}_k\mathbf{x}'_k}{\sigma^2_k\pi_k}\right)^{-1}\left(\sum_{h=1}^{H_s}\frac{n_h}{\sum_{s_h}R^{(a)}_{k|s}}\sum\nolimits_{s_h}\frac{R^{(a)}_{k|s}\mathbf{x}_k y_k}{\sigma^2_k\pi_k}\right)$$

We will focus only on the parts of the bias that are nonlinear in $R^{(a)}_{k|s}$.

They can be written as functions of estimated totals:

$$\begin{aligned}
f_1 &= \textstyle\sum_s \dfrac{\mathbf{x}_k' \hat{\mathbf{B}}_r^{(a)}}{\pi_k} \\
&= \textstyle\sum_s \dfrac{\mathbf{x}_k'}{\pi_k} \left(\displaystyle\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}_{\theta s_h}^{(a)}} \hat{\mathbf{T}}_{\mathbf{xx}s_h}^{(a)} \right)^{-1} \left(\displaystyle\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}_{\theta s_h}^{(a)}} \hat{\mathbf{t}}_{\mathbf{x}y s_h}^{(a)} \right) \\
f_2 &= \sum_{h=1}^{H_s} \frac{n_h}{\sum_{s_h} R_{k|s}^{(a)}} \textstyle\sum_{s_h} \dfrac{R_{k|s}^{(a)} y_k}{\pi_k} = \displaystyle\sum_{h=1}^{H_s} n_h \frac{\hat{t}_{y s_h}^{(a)}}{\hat{t}_{\theta s_h}^{(a)}} \\
f_3 &= -\sum_{h=1}^{H_s} \frac{n_h}{\sum_{s_h} R_{k|s}^{(a)}} \textstyle\sum_{s_h} \dfrac{R_{k|s}^{(a)} \mathbf{x}_k' \hat{\mathbf{B}}_r^{(a)}}{\pi_k} \\
&= -\left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}_{\theta s_h}^{(a)}} \hat{\mathbf{t}}_{\mathbf{x} s_h}^{(a)} \right) \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}_{\theta s_h}^{(a)}} \hat{\mathbf{T}}_{\mathbf{xx}s_h}^{(a)} \right)^{-1} \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}_{\theta s_h}^{(a)}} \hat{\mathbf{t}}_{\mathbf{x}y s_h}^{(a)} \right)
\end{aligned}$$

where $\hat{t}_{\theta s_h}^{(a)} = \sum_{s_h} R_{k|s}^{(a)}$ and $\hat{\mathbf{t}}_{\mathbf{x}s_h}^{(a)}$, $\hat{\mathbf{T}}_{\mathbf{xx}s_h}^{(a)}$ and $\hat{\mathbf{t}}_{\mathbf{x}ys_h}^{(a)}$ have the typical elements

$$\hat{t}_{js_h}^{(a)} = \textstyle\sum_{s_h} \dfrac{R_{k|s}^{(a)} x_{jk}}{\pi_k}$$

$$\hat{t}_{jj's_h}^{(a)} = \textstyle\sum_{s_h} \dfrac{R_{k|s}^{(a)} x_{jk} x_{j'k}}{\sigma_k^2 \pi_k}$$

and

$$\hat{t}_{jys_h}^{(a)} = \textstyle\sum_{s_h} \dfrac{R_{k|s}^{(a)} x_{jk} y_k}{\sigma_k^2 \pi_k}$$

Large-sample results for f_2, which is simply a sum of ratios, are well known, and results for f_1 are easily derived using the derivation for f_3, so we need only consider f_3. Using rules for differentiation (see for example chapter

15 in Harville (1997)), we get the partial derivatives:

$$
\begin{aligned}
\frac{\delta f_3}{\delta \hat{t}^{(a)}_{\theta s_h}} = & \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{t}}^{(a)}_{\mathbf{x} s_h}\right)' \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{T}}^{(a)}_{\mathbf{xx} s_h}\right)^{-1} \frac{n_h}{(\hat{t}^{(a)}_{\theta s_h})^2} \hat{\mathbf{t}}^{(a)}_{\mathbf{x}y s_h} \\
& - \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{t}}^{(a)}_{\mathbf{x} s_h}\right)' \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{T}}^{(a)}_{\mathbf{xx} s_h}\right)^{-1} \frac{n_h}{(\hat{t}^{(a)}_{\theta s_h})^2} \hat{\mathbf{T}}^{(a)}_{\mathbf{xx} s_h} \\
& \times \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{T}}^{(a)}_{\mathbf{xx} s_h}\right)^{-1} \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{t}}^{(a)}_{\mathbf{x}y s_h}\right) \\
& + \frac{n_h}{(\hat{t}^{(a)}_{\theta s_h})^2} \hat{\mathbf{t}}^{(a)}_{\mathbf{x} s_h} \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{T}}^{(a)}_{\mathbf{xx} s_h}\right)^{-1} \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{t}}^{(a)}_{\mathbf{x}y s_h}\right)
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
\frac{\delta f_3}{\delta \hat{t}^{(a)}_{jj's_h}} = & \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{t}}^{(a)}_{\mathbf{x} s_h}\right)' \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{T}}^{(a)}_{\mathbf{xx} s_h}\right)^{-1} \\
& \times \hat{\mathbf{\Lambda}}^{(a)}_{jj'h} \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{T}}^{(a)}_{\mathbf{xx} s_h}\right)^{-1} \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{t}}^{(a)}_{\mathbf{x}y s_h}\right)
\end{aligned}
\tag{B.2}
$$

$$
\frac{\delta f_3}{\delta \hat{t}^{(a)}_{jys_h}} = -\left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{t}}^{(a)}_{\mathbf{x} s_h}\right)' \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{T}}^{(a)}_{\mathbf{xx} s_h}\right)^{-1} \hat{\boldsymbol{\lambda}}^{(a)}_{jh} \tag{B.3}
$$

$$
\frac{\delta f_3}{\delta \hat{t}^{(a)}_{js_h}} = -\hat{\boldsymbol{\lambda}}'_{jh} \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{T}}^{(a)}_{\mathbf{xx} s_h}\right)^{-1} \left(\sum_{h=1}^{H_s} \frac{n_h}{\hat{t}^{(a)}_{\theta s_h}} \hat{\mathbf{t}}^{(a)}_{\mathbf{x}y s_h}\right) \tag{B.4}
$$

where $\hat{\mathbf{\Lambda}}_{jj'h}$ is a $J \times J$ matrix with $\dfrac{n_h}{\sum_{s_h} R^{(a)}_{k|s}}$ in positions (j, j') and (j', j) and the value 0 elsewhere, and $\hat{\boldsymbol{\lambda}}_{jh}$ is a J-dimensional vector with the value $\dfrac{n_h}{\sum_{s_h} R^{(a)}_{k|s}}$ in position j and zeros elsewhere.

Evaluating the partial derivatives at the expected value point (with respect to the response distribution) $(t^{(a)}_{\theta h}, \hat{\mathbf{T}}^{(a)}_{\mathbf{xx}\theta h}, \hat{\mathbf{t}}^{(a)}_{\mathbf{x}y\theta h}, \hat{\mathbf{t}}^{(a)}_{\mathbf{x}\theta h})$, and letting $\hat{\mathbf{B}}^{(a)}_{\theta} =$

$\left(\sum_{h=1}^{H_s}\frac{n_h}{t_{\theta h}^{(a)}}\hat{\mathbf{T}}_{\mathbf{xx}\theta h}^{(a)}\right)^{-1}\left(\sum_{h=1}^{H_s}\frac{n_h}{t_{\theta h}^{(a)}}\hat{\mathbf{t}}_{\mathbf{x}y\theta h}^{(a)}\right)$ we get the approximation

$$
\begin{aligned}
f_3 \doteq & \sum_{h=1}^{H_s}\frac{n_h}{t_{\theta h}^{(a)}}\hat{\mathbf{t}}_{\mathbf{x}\theta h}^{(a)}\hat{\mathbf{B}}_{\theta}^{(a)} - \sum_{h=1}^{H_s}\sum_{j}\hat{\lambda}_{jh}\hat{B}_{j\theta}^{(a)}\left(\hat{t}_{js_h}^{(a)} - \hat{t}_{j\theta h}^{(a)}\right) \\
& + \left(\sum_{h=1}^{H_s}\frac{n_h}{t_{\theta h}^{(a)}}\hat{\mathbf{t}}_{\mathbf{x}\theta h}^{(a)}\right)\left(\sum_{h=1}^{H_s}\frac{n_h}{t_{\theta h}^{(a)}}\hat{\mathbf{T}}_{\mathbf{xx}\theta h}^{(a)}\right)^{-1}\sum_{h=1}^{H_s}\sum_{j}\sum_{j'\le j}\Lambda_{jj'h}\hat{B}_{j\theta}\left(\hat{t}_{jj's_h}^{(a)} - \hat{t}_{jj'\theta h}^{(a)}\right) \\
& - \left(\sum_{h=1}^{H_s}\frac{n_h}{t_{\theta h}^{(a)}}\hat{\mathbf{t}}_{\mathbf{x}\theta h}^{(a)}\right)\left(\sum_{h=1}^{H_s}\frac{n_h}{t_{\theta h}^{(a)}}\hat{\mathbf{T}}_{\mathbf{xx}\theta h}^{(a)}\right)^{-1}\sum_{h=1}^{H_s}\sum_{j}\hat{\lambda}_{jh}\left(\hat{t}_{jys_h}^{(a)} - \hat{t}_{jy\theta h}^{(a)}\right) \\
& - \left\{\left(\sum_{h=1}^{H_s}\frac{n_h}{t_{\theta h}^{(a)}}\hat{\mathbf{t}}_{\mathbf{x}\theta h}^{(a)}\right)'\left(\sum_{h=1}^{H_s}\frac{n_h}{t_{\theta h}^{(a)}}\hat{\mathbf{T}}_{\mathbf{xx}\theta h}^{(a)}\right)^{-1}\sum_{h=1}^{H_s}\frac{n_h}{(t_{\theta h}^{(a)})^2}\left(\hat{\mathbf{T}}_{\mathbf{xx}\theta h}^{(a)}\hat{\mathbf{B}}_{\theta}^{(a)} - \hat{\mathbf{t}}_{\mathbf{x}y\theta h}^{(a)}\right)\right. \\
& \left. + \sum_{h=1}^{H_s}\frac{n_h}{(t_{\theta h}^{(a)})^2}\hat{\mathbf{t}}_{\mathbf{x}\theta h}^{(a)}\right\}\left(\hat{t}_{\theta s_h}^{(a)} - t_{\theta h}^{(a)}\right)
\end{aligned}
$$

The expectation of the above expression, with respect to the response distribution, is just

$$
E_{RD^{(a)}}(f_3) \doteq \sum_{h=1}^{H_s}\frac{n_h}{\hat{t}_{\theta h}^{(a)}}\hat{\mathbf{t}}_{\mathbf{x}\theta h}^{(a)}\hat{\mathbf{B}}_{\theta}^{(a)} = \sum_{h=1}^{H_s}\frac{n_h}{\sum_{s_h}\theta_{k|s}^{(a)}}\sum\nolimits_{s_h}\frac{\theta_{k|s}^{(a)}\mathbf{x}_k'}{\pi_k}\hat{\mathbf{B}}_{\theta}
$$

The same approximation can be made in special case 1, simply replace $\mathbf{x}$ with $\mathbf{x}_1$ above.

B.2 Nonresponse bias for the calibration estimator

The nonresponse bias for $\hat{t}_{ycal1}^{(a)}$ is given by (A.10):

$$
\begin{aligned}
E_p\Bigg[& E_{RD^{(a)}}\left(\left(\sum\nolimits_U \mathbf{x}_{1k}' - \sum\nolimits_s \frac{R_{k|s}^{(a)}\mathbf{x}_{1k}'}{\pi_k}\right)\left(\tilde{\mathbf{B}}_{1r}^{(a)} - \hat{\mathbf{B}}_{1s}\right)\right) \\
& + \sum\nolimits_s \frac{R_{k|s}^{(a)}(y_k - \mathbf{x}_{1k}'\hat{\mathbf{B}}_{1s})}{\pi_k} - \sum\nolimits_s \frac{(y_k - \mathbf{x}_{1k}'\hat{\mathbf{B}}_{1s})}{\pi_k}\Bigg]
\end{aligned}
$$

where
$$\tilde{\mathbf{B}}_{1r}^{(a)} = \left(\sum\nolimits_{r^{(a)}} \frac{c_k \mathbf{x}_{1k}\mathbf{x}'_{1k}}{\pi_k}\right)^{-1} \left(\sum\nolimits_{r^{(a)}} \frac{c_k \mathbf{x}_{1k} y_k}{\pi_k}\right)$$

The parts of the nonresponse bias that is nonlinear in $R_{k|s}^{(a)}$ can be expressed as functions of estimated totals:

$$\begin{aligned} f_1 &= \sum\nolimits_U \mathbf{x}'_{1k} \tilde{\mathbf{B}}_{1r}^{(a)} \\ &= \sum\nolimits_U \mathbf{x}'_{1k} (\tilde{\mathbf{T}}_{\mathbf{x}_1\mathbf{x}_1 s}^{(a)})^{-1} \tilde{\mathbf{t}}_{\mathbf{x}_1 y s}^{(a)} \end{aligned}$$

$$\begin{aligned} f_2 &= -\sum\nolimits_s \frac{R_{k|s}^{(a)} \mathbf{x}'_{1k} \tilde{\mathbf{B}}_{1r}^{(a)}}{\pi_k} \\ &= \tilde{\mathbf{t}}_{\mathbf{x}_1 s}^{(a)} (\tilde{\mathbf{T}}_{\mathbf{x}_1\mathbf{x}_1 s}^{(a)})^{-1} \tilde{\mathbf{t}}_{\mathbf{x}_1 y s}^{(a)} \end{aligned}$$

where $\tilde{\mathbf{t}}_{\mathbf{x}_1 s}^{(a)}$, $\tilde{\mathbf{T}}_{\mathbf{x}_1\mathbf{x}_1 s}^{(a)}$ and $\tilde{\mathbf{t}}_{\mathbf{x}_1 y s}^{(a)}$ have the typical elements

$$\tilde{t}_{js}^{(a)} = \sum\nolimits_s \frac{R_{k|s}^{(a)} \mathbf{x}'_{1k}}{\pi_k}$$

$$\tilde{t}_{jj's}^{(a)} = \sum\nolimits_s \frac{R_{k|s}^{(a)} c_k \mathbf{x}_{1jk} \mathbf{x}_{1j'k}}{\pi_k}$$

and
$$\tilde{t}_{jys}^{(a)} = \sum\nolimits_s \frac{R_{k|s}^{(a)} c_k \mathbf{x}_{1jk} y_k}{\pi_k}$$

Results for f_1 are easily derived using the derivation for f_2. The partial derivatives are

$$\frac{\delta f_2}{\delta \tilde{t}_{jj'}} = -\tilde{\mathbf{t}}_{\mathbf{x}_1 s}^{(a)} (\tilde{\mathbf{T}}_{\mathbf{x}_1\mathbf{x}_1 s}^{(a)})^{-1} \tilde{\mathbf{\Lambda}}_{jj'} (\tilde{\mathbf{T}}_{\mathbf{x}_1\mathbf{x}_1 s}^{(a)})^{-1} \tilde{\mathbf{t}}_{\mathbf{x}_1 y s}^{(a)} \tag{B.5}$$

$$\frac{\delta f_2}{\delta \tilde{t}_{jy}} = -\hat{\mathbf{t}}_{\mathbf{x}_1 s}^{(a)} (\tilde{\mathbf{T}}_{\mathbf{x}_1\mathbf{x}_1 s}^{(a)})^{-1} \tilde{\boldsymbol{\lambda}}_j \tag{B.6}$$

$$\frac{\delta f_2}{\delta \tilde{t}_j} = -\tilde{\boldsymbol{\lambda}}'_j (\tilde{\mathbf{T}}_{\mathbf{x}_1\mathbf{x}_1 s}^{(a)})^{-1} \tilde{\mathbf{t}}_{\mathbf{x}_1 y s}^{(a)} \tag{B.7}$$

where $\tilde{\mathbf{\Lambda}}_{jj'}$ is a is a $J \times J$ matrix with 1 in positions (j, j') and (j', j) and the value 0 elsewhere, and $\tilde{\boldsymbol{\lambda}}_j$ is a J-dimensional vector with the value 1 in position j and zeros elsewhere.

Evaluating the partial derivatives at the expected value point $(\tilde{\mathbf{t}}^{(a)}_{\mathbf{x}_1\theta}, \tilde{\mathbf{T}}^{(a)}_{\mathbf{x}_1\mathbf{x}_1\theta}, \tilde{\mathbf{t}}^{(a)}_{\mathbf{x}_1 y\theta})$, and letting $\tilde{\mathbf{B}}^{(a)}_{1\theta} = (\tilde{\mathbf{T}}^{(a)}_{\mathbf{x}_1\mathbf{x}_1\theta})^{-1}\tilde{\mathbf{t}}^{(a)}_{\mathbf{x}_1 y\theta}$, we get

$$
\begin{aligned}
f_2 \doteq & \textstyle\sum_s \frac{\theta^{(a)}_{k|s}\mathbf{x}'_{1k}}{\pi_k}\tilde{\mathbf{B}}^{(a)}_{1\theta} \\
& - \sum_j \tilde{\lambda}'_j \tilde{B}^{(a)}_{1j\theta}\left(\tilde{t}^{(a)}_{js} - \tilde{t}^{(a)}_{j\theta}\right) \\
& - \tilde{\mathbf{t}}_{\mathbf{x}_1\theta}(\tilde{\mathbf{T}}^{(a)}_{\mathbf{x}_1\mathbf{x}_1\theta})^{-1}\sum_j \sum_{j'\leq j} \tilde{\Lambda}^{(a)}_{jj'}\tilde{B}^{(a)}_{1j\theta}\left(\tilde{t}^{(a)}_{jj's} - \tilde{t}^{(a)}_{jj'\theta}\right) \\
& - \tilde{\mathbf{t}}^{(a)}_{\mathbf{x}_1\theta}(\tilde{\mathbf{T}}^{(a)}_{\theta})^{-1}\sum_j \tilde{\lambda}_j\left(\tilde{t}^{(a)}_{jys} - \tilde{t}^{(a)}_{jy\theta}\right)
\end{aligned}
$$

The expectation of f_2, with respect to the response distribution, is

$$
E_{RD^{(a)}}(f) \doteq \textstyle\sum_s \frac{\theta^{(a)}_{k|s}\mathbf{x}'_{1k}}{\pi_k}\tilde{\mathbf{B}}_{1\theta}
$$

The same approximation can be made for $\hat{t}^{(a)}_{ycal2}$, just replace $\mathbf{x}_1$ with $\mathbf{x}$ in the derivation above.