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A Fixed Point Theorem for Monotone Functions equivalent to Browers theorem

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Abstract

The main theorem of this paper states that if $f : R^m_+ \to R^m_+$ is increasing and continuous and the set $S = \{x \geq 0 : f(x) \leq x\}$ is bounded and contains some $x' > 0$ then there is a non-zero fixed point of f. If in addition to this there is $x'' \in S$, $x'' < x'$ there are multiple fixed points. The main theorem is equivalent to the Brouwer fixed point theorem.
Keywords: Increasing function, Fixed point

In this paper we study the existence of fixed points of continuous increasing functions defined on finite dimensional spaces. A function $f : A \to R^m$, where $A \subset R^m$, is increasing or monotone if for x' and x in A,

$$x' \geq x \Rightarrow f(x') \geq f(x). \tag{1}$$

The main result of this paper states that if $f : R^m_+ \to R^m_+$ is increasing and continuous and the set $S = \{x \geq 0 : f(x) \leq x\}$ is bounded and contains some $x' > 0$ then there is a non-zero fixed point x of f, $x = f(x) \neq 0$.

The theorem was proved, using degree theory, and applied to economic models in [1]. However, in this paper the proof uses the Knaster-Koratowski-Mazurkiewicz lemma (KKM) (see e.g. [2]), which is equivalent to the Brouwer fixed point theorem. A corollary to the theorem is added in which we obtain the Brouwer fixed point theorem. Thus, our

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theorem is equivalent to Brouwer and this shows the "non-simplicity" of our theorem, and its possible use in other applications.

Furthermore, if there are x'' and x', both in S such that $x'' < x'$, then there are at least two fixed points of f.

Theorem 1 *Assume that $f : R_+^m \to R_+^m$ is continuous and satisfies $x' \geq x \Rightarrow f(x') \geq f(x)$. Let $S = \{x \geq 0 : f(x) \leq x\}$. If S is bounded, and if there is $x' > 0$, $x' \in S$, then there is $x \geq 0$, $x \neq 0$, such that $x = f(x)$.*

Proof: For each $x' \in S$, $x' > 0$, f takes $\{x \geq 0 : x \leq x'\}$ to itself. Let $M = \text{cl}\{x \geq 0 : x' \in S, x' > 0, x \leq x'\}$, then f takes M to M. To see this, assume that there is $x \in M$ such that $f(x) \in \sim M$ which implies that there is $\epsilon > 0$ such that $|f(x) - y| > \epsilon$ for all $y \in M$. For each $\delta > 0$ there is $x'' \in M$ such that $|x - x''| < \delta$, $x'' \leq x'$, $x' \in M$ and thus $f(x'') \leq f(x') \leq x'$ and $f(x'') \in M$ which thus contradicts the continuity of f.

Let ∂M to be the boundary of M relative to R_+^m and let $\Delta = \{x \in R_+^m : \sum_i x_i = 1\}$ is the unit $(m-1)-$simplex. For $x \in \Delta$ we have that there is $\alpha > 0$ such that $\alpha x \leq x'$ for some $x' \in S$. There is also $\bar{\alpha} = \max \alpha$ for $\alpha x \in M$ since M is compact. Furthermore, $\bar{\alpha} x \in \partial M$ and $\alpha x \notin \partial M$ for $\alpha \neq \bar{\alpha}$. Then the function $h : \Delta \to \partial M$

$$h(x) = \Big\{ y \in \partial M : y\Big(\sum_i y_i\Big)^{-1} = x \Big\}$$

is well defined.

h is also continuous. Assume that h is not continuous at $x \in \Delta$. At x, $y = kx$ for some constant $k > 0$. Then, there is $\epsilon > 0$ such that for each $\delta > 0$ there is $x' \in \Delta$, $x' > 0$ such that $|x - x'| < \delta$ and $|y - y'| > \epsilon$. Let $y' = k'x'$, $k' > 0$. Then, $|x - x'| = |y/k - y'/k'| < \delta$ and since $|y - y'| > \epsilon$ we must have $y \geq y'$ or $y \leq y'$ with $y_i = y'_i$ for some i, only if $y_i = y'_i = 0$. Clearly, y and y' can not both be in ∂M which contradicts the definition of h.

The sets $C_i \subset \Delta$ defined by

$$C_i = \{x \in \Delta : f_i(h(x)) \geq h_i(x)\}, \quad i = 1, \ldots, m$$

is a closed covering of Δ. If $x \notin C_i$ for all i, then $f(h(x) < h(x)$ contradicting the definition of ∂M. For each C_i we have that $C_i \supset \Delta^i = \{x \in \Delta : x_i = 0\}$. By the KKM lemma there is $x \in \cap_i C_i$, and then $f(y) \geq y$, $y = h(x)$. Continuity of f and the definition of M then yield $y = f(y)$. Q.E.D.

Corollary 2 *The Brouwer Fixed Point Theorem. Let $f : \Delta \to \Delta$ be continuous, where $\Delta = \{x \in R_+^m : \sum_i x_i = 1\}$ is the unit $(m-1)-$simplex. Then, there is $x \in \Delta$ such that $x = f(x)$.*

Proof: Define $g : R_+^m \to R_+^m$ by

$$g_i(x) = \begin{cases} \inf\{f_i(x') : x' \in \Delta, x \le x'\} & \text{if } \sum_j x_j \le 1 \\ \sup\{f_i(x') : x' \in \Delta, x \ge x'\} & \text{if } \sum_j x_j \ge 1 \end{cases} \quad (2)$$

Then, g is continuous and monotone. Finally, define $h : R_+^m \to R_+^m$ by

$$h(x) = \left(\sum_j x_j\right)^\alpha g(x)$$

which implies that h satisfies the conditions of Theorem 1 for $\alpha > 1$ sufficiently large. Then there is $\bar{x} \neq 0$ such that $\bar{x} = h(\bar{x})$ or

$$\bar{x} = \left(\sum_j \bar{x}_j\right)^\alpha g(\bar{x})$$

Assume next that $\sum_j \bar{x}_j > 1$. From the definition of g we have that $g(\bar{x}) \geq f(x')$ for some $x' \in \Delta$ and thus $\sum_i g_i(\bar{x}) \geq 1$ and

$$\sum_i \bar{x}_i < \left(\sum_j \bar{x}_j\right)^\alpha \sum_i g_i(\bar{x})$$

which contradicts $\bar{x}$ being a fixed point. Thus, $0 < \sum_j \bar{x}_j \leq 1$. Due to this we find that $\bar{x} \leq g(\bar{x})$. $\Delta' = \{x \geq \bar{x} : \sum_i x_i = 1\}$ is a proper subsimplex of Δ. For $x \in \Delta'$ and the monotonicity of g we have that $g(x) \geq g(\bar{x})$. Furthermore, from the definition of g we have $f(x) \geq g(x)$. Combining all inequalities we find that $f(x) \geq \bar{x}$. Thus,

$$f(\Delta') \subset \Delta'$$

Define X to be the set of vectors of all monotone sequences $\{x^n\}$, $x^{n+1} \geq x^n$ such that $x^0 \geq \bar{x}$ and $\sum_i x_i^n \leq 1$. Specifically, the elements of the sequence defined by $x^0 = \bar{x}$, $x^{n+1} = g(x^n)$ is in this set, and since g is continuous, this sequence converges to some $x^* = g(x^*)$. From Zorn's lemma (see e.g. [3]) it follows that there is $y \in X$ such that y is greater than all elements of X. Clearly, $y \in \Delta'$ and thus $f(y) \in \Delta'$ and from the definition of g it follows that $x^* \leq g(y) = f(y)$ and hence, $f(y) \in X$, and $f(y) \leq y$ which implies that $y = f(y)$. Q.E.D.

The following proposition deals with functions which map the whole of R^m into itself. The result is a consequence of Tarski's fixed point theorem (see e.g. [4]) and it is easy to prove directly as in [5].

Proposition 3 *Assume that $f : R^m \rightarrow R^m$ is continuous and satisfies $x' \geq x \Rightarrow f(x') \geq f(x)$. Let $S = \{x : f(x) \leq x\}$ be non–empty and bounded from below. Then, for each $x'' \in S$, there is $\bar{x} \leq x''$ such that $\bar{x} = f(\bar{x})$.*

We can now combine the results of these propositions to obtain conditions for the existence of more than one fixed point.

Proposition 4 *Assume that $f : R^m \rightarrow R^m$ is continuous and satisfies $x' \geq x \Rightarrow f(x') \geq f(x)$. Let $S = \{x \geq 0 : f(x) \leq x\}$. Assume that S is bounded and contains two points x' and x'' such that $x'' < x'$. Then, there are two fixed points, $\bar{x}$ and $\tilde{x}$, of f for which $\bar{x} \leq \tilde{x}$ ($\bar{x} \neq \tilde{x}$).*

Proof: The fixed point $\bar{x}$ is given by a direct application of Proposition 3. Next, set $z = x - \bar{x}$ which yields $z' = x' - x > 0$. Define $g(z) = f(x) - \bar{x}$. Due to monotonicity of f, $g(z)$ maps R_+^m into itself and by assumption the set $S = \{z \geq 0 : g(z) \leq z\}$ contains $z' > 0$. Theorem 1 can now be applied and we thus have a fixed point $\tilde{z} = g(\tilde{z}) \geq 0,\ \tilde{z} \neq 0$, i.e. $\tilde{x} - \bar{x} = f(\tilde{x}) - \bar{x} \geq 0,\ \tilde{x} \neq \bar{x}$. Q.E.D.

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