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Statistical Inference for the Tangency Portfolio in High Dimension

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Abstract

In this paper, we study the distributional properties of the tangency portfolio (TP) weights assuming a normal distribution of the logarithmic returns. We derive a stochastic representation of the TP weights that fully describes their distribution. Under a high-dimensional asymptotic regime, i.e. the dimension of the portfolio, k, and the sample size, n, approach infinity such that $k/n \to c \in (0, 1)$, we deliver the asymptotic distribution of the TP weights. Moreover, we consider tests about the elements of the TP and derive the asymptotic distribution of the test statistic under the null and alternative hypotheses. In a simulation study, we compare the asymptotic distribution of the TP weights with the exact finite sample density. We also compare the high-dimensional asymptotic test with an exact small sample test. We document a good performance of the asymptotic approximations except for small sample sizes combined with c close to one. In an empirical study, we analyze the TP weights in portfolios containing stocks from the S&P 500 index.

MSC: 62H10, 62H12, 91G10.

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1 Introduction

The fundamental goal of the portfolio theory introduced by Markowitz (1952) is to efficiently allocate investments among various assets. The mean-variance optimization technique serves as a quantitative tool that considers the trade-off between the risk of the portfolio and its return. In the formulation of the mean-variance analysis the investor selects a portfolio with the highest expected return for a given level of risk or the smallest risk for a given level of the expected return. The risk aversion strategy in the absence of risk-free assets (bonds) leads to the minimum variance portfolio, whereas in the presence of risk-free assets, the tangency portfolio (TP) is constructed and it consists of both risky and risk-free assets. Moreover, it is the only portfolio that maximizes the Sharpe ratio. Because of its significant role in finance for both researchers and practitioners, having a full understanding of the properties of the TP becomes vital for any financial actor.

The statistical properties of the estimated TP weights are investigated in a number of papers. Britten-Jones (1999) developed an exact finite sample F-test for TP weights. Okhrin and Schmid (2006), under the assumption of independently and multivariate normally distributed returns, derived the univariate density of the TP weights as well as its asymptotic distribution. Bodnar (2009) proposed a sequential monitoring procedures for the TP weights, while Bodnar and Okhrin (2011) suggested several exact test of general linear hypotheses about the elements of the portfolio weights. In Kotsiuba and Mazur (2015), the asymptotic distribution and the approximate density function, based on a third order Taylor series approximation, of the TP weights are derived. Bodnar and Zabolotskyy (2017) considered the risk properties of the TP and concluded that this portfolio is a very risky investment opportunity which should be carefully considered in practice. Bauder et al. (2018) studied different distributional properties of TP weights from Bayesian statistics point of view. Bodnar et al. (2019b) analyzed the distributional properties of the estimated TP weights and proposed inference procedures in small and high dimensions when both the population and the sample covariance matrices are singular. A test of the existence of TP on the set of feasible portfolios is developed by Muhinyuza et al. (2020) which is also extended to the high-dimensional setting by Muhinyuza (2020). Higher-order moments of the estimated TP weights are obtained by Javed et al. (2020), while Alfelt and Mazur (2020) studied the mean and variance of the estimated TP weights for small samples.

The present paper complements the existing literature by delivering the stochastic representation and asymptotic distribution of the estimated TP weights as well as the asymptotic distribution of the statistical test about the elements of the TP. Asymptotic results are delivered under a high-dimensional asymptotic regime, i.e. $k/n \to c \in (0, 1)$ as $k \to \infty$ and $n \to \infty$, and assuming positive definiteness of the population covariance matrix.

The remaining parts of this paper are organized as follow. In Section 2, we present a very useful stochastic representation of the estimated TP weights that depicts their distribution. The obtained stochastic representation is then used in the derivation of the high-dimensional asymptotic distribution of the estimated TP weights and high-dimensional asymptotic test on the TP weights. In Section 3, we present the results of the simulation and empirical studies, while the summary and concluding remarks are given in Section 4. All proofs are collected in the appendix.

2 Main Results

We consider a portfolio that consists of k risky assets. Let $\mathbf{x}_t = (x_{1t}, \ldots, x_{kt})^T$ be the k-dimensional vector of log-returns of these assets at time point $t = 1, \ldots, n$, and $\mathbf{w} = (w_1, \ldots, w_k)^T$ be a vector of weights, where w_i denotes the portion of the wealth allocated to the ith asset. Let also the mean vector of the asset returns be denoted by $\boldsymbol{\mu}$ and the covariance matrix by $\boldsymbol{\Sigma}$ which assumed to be positive definite. Following the mean-variance theory introduced by Markowitz (1952), an investor allocates her/his wealth among k risky assets by maximizing the portfolio expected return for a given level of the portfolio risk or, equivalently, by minimizing the risk given some predetermined level of the portfolio expected return. In this context, the risk is measured by the variance of the portfolio return. Levy and Markowitz (1979) and Kroll et al. (1984) showed that the mean-variance portfolio problem is equivalent to maximizing the expected quadratic utility. In the absence of a risk-free asset, the optimal portfolio is obtained by maximizing the expected quadratic utility under the constraint $\mathbf{w}^T\mathbf{j}_k = 1$, where $\mathbf{j}_k$ denotes the vector of ones. On the other hand, if short selling is allowed and there is a possibility to invest in the risk free-asset with return r_f, a portion of an investor's wealth may be invested in the risk-free asset and it may reduce the variance, while the rest of the wealth can be invested in the risky assets. In this case, the expected return of the portfolio is given by $\mu_p = \mathbf{w}^T(\boldsymbol{\mu} - r_f\mathbf{j}_k) + r_f$ with the variance $\sigma_p^2 = \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$. The optimal portfolio composition of the tangency portfolio (TP) is obtained by solving the following optimization problem

$$\mu_p - \frac{\alpha}{2}\sigma_p^2 \to \max_{\mathbf{w}} \tag{1}$$

where the coefficient α describes the investor's attitude towards risk or risk aversion. All portfolios from the tangent line are obtained by varying $\alpha \in (0, \infty)$. The higher value of the risk aversion representing lesser tolerance to risk. The risk aversion level can be looked as a characteristic of the investor's indifference curve which represents the investor's preference for risk and return. How to choose or fix the value of α in practice is not obvious and a number of papers have suggested different approaches to estimating the risk aversion coefficient (see, e.g.,Chetty (2003); Campo et al. (2011); Bodnar and Okhrin (2013); Bodnar et al. (2018b)).

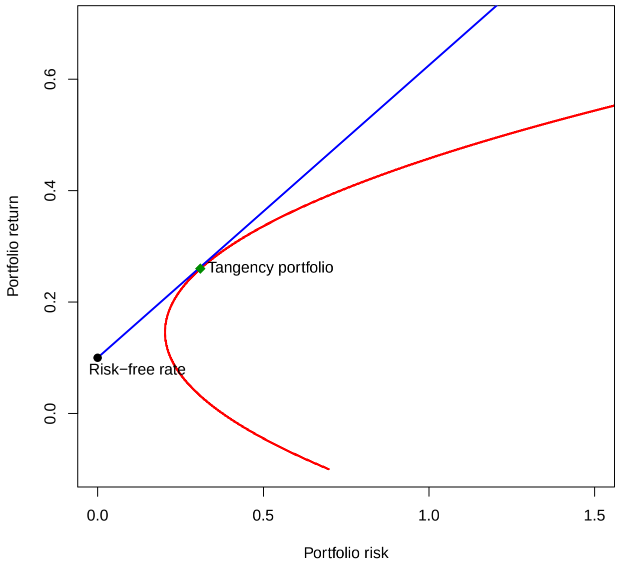


Figure 1: A graphical illustration of the efficient frontier in the presence of risk-free asset.

When solving the maximization problem defined in (1), we note that short sales are allowed and there are no restrictions on the portfolio weights, therefore, the optimization problem is unconstrained. Consequently, it is easy to see that the global maximum, i.e. the TP weights, is given by

$$\mathbf{w}_{TP} = \alpha^{-1}\mathbf{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{j}_k\right). \tag{2}$$

Equation (2) gives the structure of the optimal portfolio composition corresponding to the risky assets only, whereas the portion invested into the risk-free asset is determined by $1-\mathbf{w}_{TP}^T\mathbf{j}_k$. Ingersoll (1987) defined a TP as a tangent point which lies on the intersection of the mean-variance frontier and the tangency line drawn from the return of the risk-free asset (see Figure 1).

The optimal portfolio weights depend on the unknown parameters $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ and in practice they need to be estimated. Using the random sample we estimate the parameters by their sample counterparts as

$$\overline{\mathbf{x}} = \frac{1}{n}\sum_{t=1}^{n}\mathbf{x}_t \qquad \text{and} \qquad \mathbf{S} = \frac{1}{n-1}\sum_{t=1}^{n}\left(\mathbf{x}_t - \overline{\mathbf{x}}\right)\left(\mathbf{x}_t - \overline{\mathbf{x}}\right)^T.$$

Replacing $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ with $\overline{\mathbf{x}}$ and $\mathbf{S}$ in (2), we get the sample estimator of the TP weights given by

$$\hat{\mathbf{w}}_{TP} = \alpha^{-1}\mathbf{S}^{-1}\left(\overline{\mathbf{x}} - r_f\mathbf{j}_k\right). \tag{3}$$

In practice, interest is often focused on just a few weights. In addition, analysis of the whole vector becomes impractical as the dimensions k increases. We will hence focus on the linear combination of $\mathbf{w}_{TP}$ that is given by

$$\theta = \mathbf{l}^T\mathbf{w}_{TP} = \alpha^{-1}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{j}_k\right),$$

where $\mathbf{l}$ is a k-dimensional vector of constants. Consequently, the sample estimator of θ is expressed as

$$\hat{\theta} = \mathbf{l}^T\hat{\mathbf{w}}_{TP} = \alpha^{-1}\mathbf{l}^T\mathbf{S}^{-1}\left(\overline{\mathbf{x}} - r_f\mathbf{j}_k\right).$$

By choosing different vectors $\mathbf{l}$ we are able to provide information about different linear combinations of the TP weights and more insights into the behaviour of the TP. For example, by choosing $\mathbf{l} = (1, 0, \ldots, 0)^T$, an investor is able to study the behaviour of the first asset in the portfolio. Similarly, if $\mathbf{l} = (1, 1, 0, \ldots, 0)^T$ an investor is interested in the behaviour of the TP weights the two first assets of the portfolio. Taking $\mathbf{l} = \mathbf{j}_k$ an investor can study the share of the portfolio invested in risky assets.

In the following proposition, we derive a stochastic representation of $\hat{\theta}$. The stochastic representation is a powerful tool in the theory of multivariate statistics, it can be used to determine the distribution of random quantity as the distribution of functions of independent random variables with the standard probability distributions. It also plays an important role in both frequentist and Bayesian statistics (see Givens and Hoeting (2012), Bodnar et al. (2017a), Bauder et al. (2018)). Its usefulness is frequently remarkable in Monte Carlo simulations (Givens and Hoeting (2012)) as well as in elliptical contoured distributions (Gupta et al. (2013)).

Proposition 1. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be identically and independently distributed random vectors with $\mathbf{x}_1 \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $k < n$. Also, let $\mathbf{l}$ be a k-dimensional vector of constants. Then the stochastic representation of $\hat{\theta} = \mathbf{l}^T\hat{\mathbf{w}}_{TP}$ is given by*

$$\hat{\theta} \stackrel{d}{=} \frac{n-1}{\xi}\left(\theta + \alpha^{-1}z_0\sqrt{\left(\frac{1}{n} + \frac{k-1}{n(n-k+1)}u\right)\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}}\right),$$

where $\xi \sim \chi^2_{n-k}$, $z_0 \sim \mathcal{N}(0,1)$ and $u \sim \mathcal{F}(k-1, n-k+1, ns)$ with $s = (\boldsymbol{\mu} - r_f\mathbf{j}_k)^T\mathbf{R}_\mathbf{l}(\boldsymbol{\mu} - r_f\mathbf{j}_k)$ and $\mathbf{R}_\mathbf{l} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{l}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}/\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}$. Moreover, the random variables ξ, z_0 and u are mutually independently distributed.

The proof of Proposition 1 can be found in the appendix. From Proposition 1, we have a stochastic representation of $\hat{\theta}$ as a function of independently distributed χ^2, standard normal and non-central $\mathcal{F}$ random variables. It is worth noting that the application of Proposition 1 speeds up the simulation of $\hat{\theta}$, since it is sufficient to simulate only three univariate random quantities instead of generating a sample mean vector and sample covariance matrix that can have high dimensions. Proposition 1 also plays an important role in the derivation of the distribution of the linear combination of the estimated TP weights in high dimensions.

Remark 1. According to (3), the sample estimator of the TP weights $\hat{\mathbf{w}}_{TP}$ depends on the inverse of the sample covariance matrix $\mathbf{S}$. In Proposition 1, it is assumed that $k < n$ and this assumption ensures that the distribution of $\mathbf{S}$ is non-singular, therefore, the regular inverse of $\mathbf{S}$ can be taken. If $k > n$, the distribution of $\mathbf{S}$ is singular and the regular inverse cannot be used. This issue is discussed in the portfolio context by utilizing Moore-Penrose inverse (see Bodnar et al. (2016, 2017b), Tsukuma (2016), Bodnar et al. (2019b)). Alternatively, one can use different regularization techniques such as the ridge-type approach (Tikhonov and Arsenin (1977)), the Landweber Fridman iterative algorithm (Kress (1999)), the spectral cut-off (Chernousova and Golubev (2014)), a form of Lasso (Brodie et al. (2009)), and an iterative method based on a second order damped dynamical systems (Gulliksson and Mazur (2019)).

Remark 2. In the Bayesian setting, the posterior distribution of the covariance matrix $\mathbf{\Sigma}$ is inverse Wishart. Consequently, the posterior distribution of $\mathbf{w}_{TP}$ can be expressed as the product of the (singular) Wishart matrix and a normal vector. The distributional properties of this product are well studied by Bodnar et al. (2013, 2014), Bodnar et al. (2018a), Bodnar et al. (2019a).

Next, we study the asymptotic behaviour of $\hat{\theta} = \mathbf{l}^T\hat{\mathbf{w}}_{TP}$ under a high-dimensional asymptotic regime, that is, the portfolio size k increases together with the sample sizes n and they all tend to infinity. More precisely, we assume that $k_n \equiv k(n)$ and $c_n := k_n/n \to c \in (0,1)$ as $k \to \infty$ and $n \to \infty$. The following condition is needed for ensuring the validity of the asymptotic results presented in this section:

(A1) there exists m and M such that $0 < m \le \boldsymbol{\mu}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu} \le M < \infty$, $0 < m \le \mathbf{j}^T\mathbf{\Sigma}^{-1}\mathbf{j} \le M < \infty$ and $0 < m \le \mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l} \le M < \infty$ uniformly in k.

Let us note that we don't have assumptions about the eigenvalues of the population covariance matrix $\mathbf{\Sigma}$. Consequently, one can consider the case when $\mathbf{\Sigma}$ has unbounded spectrum.

In the next theorem we deliver the high-dimensional asymptotic distribution of a linear combination of the estimated TP weights for normally distributed data.

Theorem 1. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be identically and independently distributed random vectors with $\mathbf{x}_1 \sim \mathcal{N}_k(\boldsymbol{\mu}, \mathbf{\Sigma})$, $k < n$. Let $c_n := k_n/n \to c \in (0,1)$ as $n \to \infty$. Also, let $\mathbf{l}$ be a k-dimensional vector of constants. Then, under (A1), it holds that the asymptotic distribution of $\hat{\theta} = \mathbf{l}^T\hat{\mathbf{w}}_{TP}$ is given by*

$$\sqrt{n-k_n}\,\sigma^{-1}\left(\hat{\theta} - \frac{n-1}{n-k_n}\theta\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

where

$$\sigma^2 = \frac{\alpha^{-2}}{(1-c)^2}\left[\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l} + (\alpha\theta)^2 + \mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}(\boldsymbol{\mu} - r_f\mathbf{j}_k)^T\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{j}_k)\right].$$

The proof of Theorem 1 is provided in the appendix. From Theorem 1, we can observe that the sample estimator of θ is biased and, therefore, bias correction can be applied. In Corollary 1, we construct an unbiased estimator of θ and deliver its central limit theorem in the high-dimensional setting. The statement of the corollary follows immediately from Theorem 1.

Corollary 1. *Let $\widetilde{\theta} = \frac{n-k_n}{n-1}\hat{\theta}$. Under the assumptions of Theorem 1, $\widetilde{\theta}$ is asymptotically unbiased with asymptotic distribution*

$$\sqrt{n-k_n}\breve{\sigma}^{-1}\left(\widetilde{\theta} - \theta\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

where

$$\breve{\sigma}^2 = \alpha^{-2}\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l} + \theta^2 + \alpha^{-2}\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}(\boldsymbol{\mu} - r_f\mathbf{j}_k)^T\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{j}_k).$$

Having established the asymptotic distribution we next consider testing the hypothesis

$$H_0 : \mathbf{l}^T\mathbf{w}_{TP} = 0 \qquad \text{against} \qquad H_1 : \mathbf{l}^T\mathbf{w}_{TP} = \rho \neq 0. \tag{4}$$

in a high dimensional setting. Bodnar and Okhrin (2011) suggested the following test statistics for (4)

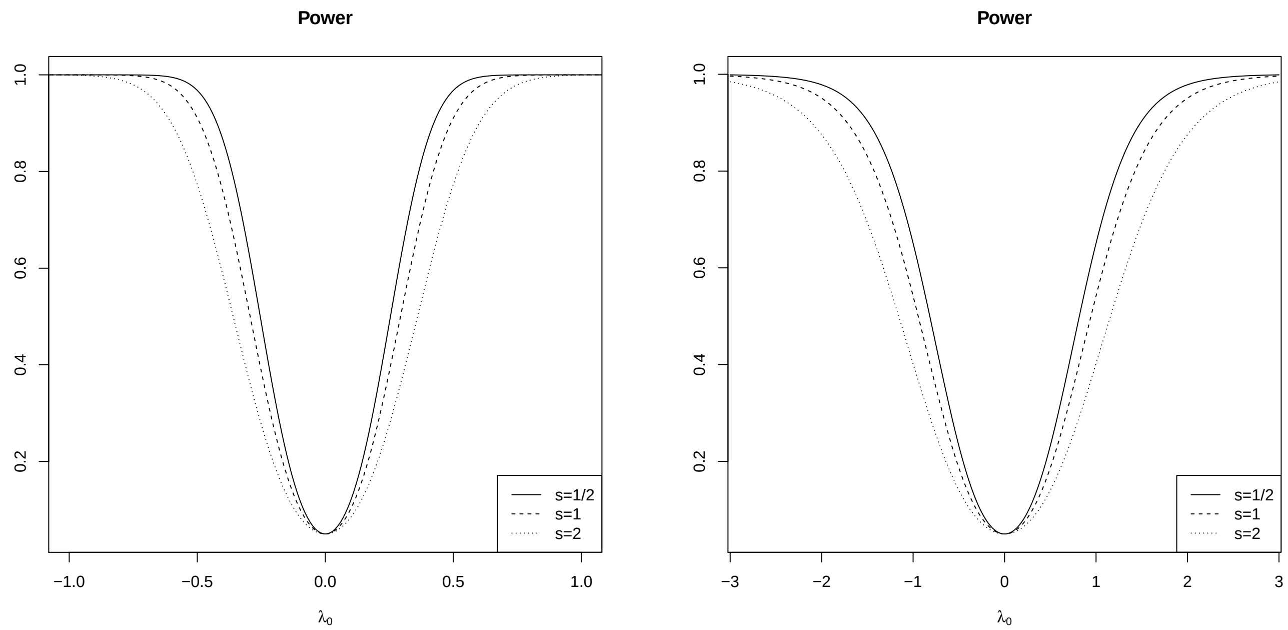
$$T = \sqrt{\frac{n(n-k_n)}{n-1}}\frac{\alpha\hat{\theta}}{\sqrt{\mathbf{l}^T\mathbf{S}^{-1}\mathbf{l}}\sqrt{1+\frac{n}{n-1}\hat{s}}},$$

where $\hat{s} = (\overline{\mathbf{x}} - r_f\mathbf{j}_k)^T\hat{\mathbf{R}}_\mathbf{l}(\overline{\mathbf{x}} - r_f\mathbf{j}_k)$ and $\hat{\mathbf{R}}_\mathbf{l} = \mathbf{S}^{-1} - \mathbf{S}^{-1}\mathbf{l}\mathbf{l}^T\mathbf{S}^{-1}/\mathbf{l}^T\mathbf{S}^{-1}\mathbf{l}$. Moreover, they delivered the distribution of T both under the null and under alternative hypotheses.

It follows from Bodnar and Okhrin (2011, Theorem 6) that the power of the test is given by

$$\begin{aligned} G_{T,\psi}(\lambda, s) &= 1 - \frac{n(n-k+1)}{(k-1)(n-1)}\int_0^\infty \left(F_{t_{n-k,v(\lambda,y)}}(t_{n-k;1-\psi/2}) - F_{t_{n-k,v(\lambda,y)}}(t_{n-k;\psi/2})\right) \\ &\quad \times f_{\mathcal{F}_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)}y\right)dy \end{aligned}$$

where ψ denotes the size of the test, $\lambda = \alpha\rho/\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}}$, $s = (\boldsymbol{\mu} - r_f\mathbf{j}_k)^T\mathbf{R}_\mathbf{l}(\boldsymbol{\mu} - r_f\mathbf{j}_k)$, $t_{n-k,v(\lambda,y)}$ stands for a non-central t-distribution with $n-k$ degrees of freedom and non-centrality parameter $v(\lambda, y) = \lambda/\sqrt{1/n + y/(n-1)}$, while $t_{n-k,\psi}$ denotes the ψ-quantile



(a) $k = 10$, $n = 100$ (b) $k = 90$, $n = 100$

Figure 2: Power of the test statistic T as a function of λ for $s \in \{1/2, 1, 2\}$, $n = 100$ and $k \in \{10, 90\}$.

of the central t-distribution with $n - k$ degrees of freedom. The power of the test thus only depends on the true alternative, ρ, and the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ through λ and s where we can think of λ as the standardized alternative and s as the slope of the efficient frontier.

In Figure 2, we illustrate the behaviour of the power of the test statistic T as a function of λ for fixed values of $s \in \{1/2, 1, 2\}$. We consider the sample size to be $n = 100$ and portfolio size to be $k \in \{10, 90\}$. We observe that the power of the test increase as s decreases. We also note that the test rejects the null hypothesis for small values of λ.

The theorem below gives us the distribution of the test statistics T in a high dimensional setting, while its proof can be found in the appendix.

Theorem 2. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be identically and independently distributed random vectors with $\mathbf{x}_1 \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $k < n$. Let $c_n := k_n/n \to c \in (0, 1)$ as $n \to \infty$. Also, let $\mathbf{l}$ be a k-dimensional vector of constants. Then, under (A1), it holds that the asymptotic distribution of T is given by*

(a)

$$\sigma_T^{-1} \left(T - \frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T \boldsymbol{\Sigma}^{-1} \mathbf{l} \left(1 + \frac{k_n - 1}{n - k_n + 1}\left(1 + \frac{n}{k_n - 1} s\right)\right)}} \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$$

where

$$\sigma_T^2 = 1 + \frac{(\alpha\rho)^2}{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}(1+s)}\left(\frac{1}{2} + \frac{s^2+c+2s}{2(1+s)^2}\right)$$

with $s = (\boldsymbol{\mu} - r_f\mathbf{j}_k)^T\mathbf{R}_\mathbf{l}(\boldsymbol{\mu} - r_f\mathbf{j}_k)$ *and* $\mathbf{R}_\mathbf{l} = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{l}\mathbf{l}^T\mathbf{\Sigma}^{-1}/\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}$*;*

(b) under the null hypothesis it holds that $T \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$*.*

3 Simulation and Empirical Studies

3.1 Simulation Study

In this subsection we present the results of a Monte Carlo simulation study. We investigate the performance of the high-dimensional asymptotic distribution of a linear combination of the TP weights derived in Theorem 1, and the power function of the high-dimensional asymptotic test that is obtained in Theorem 2.

We set $\alpha = 3$, $r_f = 0.005$, and $\mathbf{l} = (1, 0, \ldots, 0)^T$. Each element of $\boldsymbol{\mu}$ is generated from the uniform distribution on $[-0.1, 0.1]$. The population covariance matrix is drawn as follow:

- k non-zero eigenvalues of $\mathbf{\Sigma}$ are generated from the uniform distribution on (0,1);
- the eigenvectors are generated from the Haar distribution by simulating a Wishart matrix with 30 degrees of freedom and identity covariance, and calculating its eigenvectors.

Both the mean vector and the population covariance matrix obtained in this manner satisfy assumption (A1), they are then used in all simulation runs.

First, we compare the asymptotic normal distribution of $\hat{\theta} = \mathbf{l}^T\hat{\mathbf{w}}_{TP}$ with the corresponding finite-sample one obtained by applying the stochastic representation obtained in Proposition 1. We consider different sample size $n \in \{50, 120, 250, 500\}$ which roughly corresponds to the length of one-year, two-years, five-years and ten-years of weekly financial data. The results are compared for different values of concentration coefficients $c \in \{0.1, 0.4, 0.7, 0.9\}$ and it is based on $N = 10^5$ independent realisations of $\hat{\theta}$ generated from the finite-sample distribution. Lastly, the corresponding kernel density estimator of the finite sample density is computed with Epanechnikov kernel. The following algorithm is used in drawing the finite-sample density

a) generate $\hat{\theta}$ by using the stochastic representation given in Proposition 1

$$\hat{\theta} \overset{d}{=} \frac{n-1}{\xi}\left(\theta + \alpha^{-1}\sqrt{\left(\frac{1}{n} + \frac{k_n - 1}{n(n-k_n+1)}u\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}}\,z_0\right)$$

where $\xi \sim \chi^2_{n-k_n}$, $z_0 \sim \mathcal{N}(0,1)$ and $u \sim \mathcal{F}(k_n-1, n-k_n+1, ns)$ with $s = (\boldsymbol{\mu} - r_f \mathbf{j}_k)^T \mathbf{R}_\mathbf{l} (\boldsymbol{\mu} - r_f \mathbf{j}_k)$ and $\mathbf{R}_\mathbf{l} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{l}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}/\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}$; moreover, the random variables ξ, z_0 and u are mutually independently distributed;

b) compute

$$\sqrt{n-k_n}\sigma^{-1}\left(\hat{\theta} - \frac{n-1}{n-k_n}\theta\right) \tag{5}$$

with

$$\sigma^2 = \frac{\alpha^{-2}}{(1-c)^2}\left[\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l} + (\alpha\theta)^2 + \mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}(\boldsymbol{\mu} - r_f\mathbf{j}_k)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{j}_k)\right];$$

c) repeat a)-b) N times.

In Figures 3-6, we present the results of the simulation study for $c \in \{0.1, 0.4, 0.7, 0.9\}$, respectively. The finite-sample distribution of (5) is shown as dashed black lines, while the asymptotic distribution (standard normal) is shown as solid black lines. We observe that all obtained results show a good performance of the asymptotic approximation except for $c = 0.9$ and $n = 50$ where the approximation performs badly. It is can be noted that even for $n = 50$ and $c \in \{0.1, 0.4, 0.7\}$ our asymptotic results seem to provide a reasonable approximation.

From Theorem 1, we have that $\hat{\theta}$ is a biased estimator of θ. In Table 1, we study behaviour of the relative bias, $E(\hat{\theta} - \theta)/\theta$, for different values of the sample size $n \in \{50, 120, 250, 500\}$ and concentration ratio $c \in \{0.1, 0.4, 0.7, 0.9\}$. The relative bias is increasing in c and converges to $c/(1-c)$ as n increases, confirming the result of Theorem 1 and indirectly showing that the bias correction of Corollary 1 works. It is also noteworthy that the convergence is from above, i.e. the small sample bias is greater than the asymptotic bias.

n	$c = 0.1$	$c = 0.4$	$c = 0.7$	$c = 0.9$
50	0.1472	0.7377	2.7638	14.7060
120	0.1240	0.6933	2.5012	11.1048
250	0.1173	0.6849	2.4068	9.8687
500	0.1140	0.6746	2.3723	9.3402
∞, $c/(1-c)$	0.1111	0.6667	2.3333	9.0000

Table 1: Relative bias of $\hat{\theta}$ for different values of the sample size $n \in \{50, 120, 250, 500\}$ and concentration ratio $c \in \{0.1, 0.4, 0.7, 0.9\}$.

The second part of our simulation study compares the exact test with the high-dimensional asymptotic test that is derived in Theorem 2. Table 2 reports on the empirical size of the exact and asymptotic tests for a nominal significance level of 5%. As can be expected, the exact test has the correct size in all cases. The overall size properties of the asymptotic test are also quite good although the test is oversized for large c and small n.

Figures 7-10 show the power of the exact and asymptotic tests for different values of c and n as a function of λ with $s = 1$ and 5% nominal significance. For large n and/or small c the power is almost indistinguishable while the asymptotic test shows larger power than the exact test for small n and large c reflecting the oversized nature of the test.

$c \backslash n$	50		120		250		500	
	Exact	Asymptotic	Exact	Asymptotic	Exact	Asymptotic	Exact	Asymptotic
0.1	0.0512	0.0576	0.0503	0.0531	0.0504	0.0517	0.0497	0.0503
0.4	0.0504	0.0596	0.0502	0.0540	0.0505	0.0524	0.0506	0.0514
0.7	0.0508	0.0700	0.0502	0.0580	0.0503	0.0540	0.0510	0.0529
0.9	0.0505	0.1083	0.0496	0.0734	0.0508	0.0618	0.0504	0.0556

Table 2: Empirical size of the exact and asymptotic tests at 5% significance level

3.2 Empirical study

In this part, we present the results of an empirical study in which we show how the theoretical results obtained in Section 2 can be applied to real data. We consider the weekly averages of the daily log returns data from S&P 500 of 270 stocks for the period from January 3, 2007, to December 27, 2017, making a total of 574 observations. We use the weekly returns of the three-moths US treasury bill as the risk free-rate, and the risk aversion coefficient is chosen to be 3. We choose to use weekly logarithmic returns because they can be well approximated by Gaussian distribution (see Fama (1976), Tu and Zhou (2004)).

In Figures 11-14, we present the dynamic behaviour of the p-values obtained from the exact and the asymptotic tests on the hypotheses (4), specifically testing the hypothesis that the weight of one stock is zero with $\mathbf{l}$ a vector of zeros except for one element set to one, by using a rolling window estimator with an estimation window of 300 weeks, i.e. $n = 300$. We analyze portfolios with different number of assets such that $c \in \{0.1, 0.4, 0.7, 0.9\}$, i.e. $k \in \{30, 120, 210, 270\}$. The figures present the results for four stocks, Abbott Laboratories, Affiliated Managers Group Inc, Alphabet Inc Class A, and 3M Company, where we test – in turn – that the portfolio weight is zero. First of all, we would note that the p-values obtained from both tests are indistinguishable indicating that the high-dimensional asymptotic test performs well. Next, we can observe that in most cases the obtained p-values are relatively large resulting in the conclusion that the null hypothesis (4) cannot be rejected. However, the TP weights are significantly different from zero for all considered stocks from the end of 2012 until the middle of 2014 for the small portfolio size $k = 30$. For Abbott Laboratories and Affiliated Managers Group Inc the TP weights are also significant well into 2017 and 2016, respectively. For larger k and hence larger investment universes we find few occasions with significant TP weights. This is hardly surprising for two reasons. With larger k and fixed sample size c increases and we can expect lower power. In addition as the investment universe increases the true TP weights will, on average across stocks, be smaller.

4 Conclusion

This paper discussed the statistical properties of the TP weights in high dimension. In particular, we delivered the high-dimensional asymptotic distribution of the weights as well as the high-dimensional asymptotic test on the weights. All theoretical results are obtained under the assumption of normality and they can be extended to the more general case which deserves a separate study. In particular, we are planning to develop new techniques in random matrix theory that will be used for delivering a high-dimensional theory on the weights with more general distributional assumptions. In future research, we would also extend our results to the case when $c > 1$. This case is more complicated since the weights will depend on the inverse of the singular sample covariance matrix.

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Appendix

In this section, we collect all the proofs of the main results delivered in Section 2.

Proof of Proposition 1. From Theorem 3.1.2 of Muirhead (1982), it follows that

$$\overline{\mathbf{x}} \sim \mathcal{N}_k\left(\boldsymbol{\mu}, \frac{1}{n}\mathbf{\Sigma}\right) \qquad \text{and} \qquad (n-1)\mathbf{S} \sim \mathcal{W}_k(n-1, \mathbf{\Sigma}),$$

where $\mathcal{W}_k(n-1, \mathbf{\Sigma})$ denotes a k-dimensional Wishart distribution with $n-1$ degrees of freedom and the parameter matrix $\mathbf{\Sigma}$. Moreover, $\overline{\mathbf{x}}$ and $(n-1)\mathbf{S}$ are independent. Applying Corollary 1 of Bodnar and Okhrin (2011), we get the statement of the proposition. □

Proof of Theorem 1. Using the stochastic representation obtained in Proposition 1, we get

$$\begin{aligned}\hat{\theta}-\frac{n-1}{n-k_n}\theta &= \frac{n-1}{\xi}\theta+\alpha^{-1}\frac{n-1}{\xi}\frac{z_0}{\sqrt{n}}\sqrt{\left(1+\frac{k_n-1}{n-k_n+1}u\right)\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}}-\frac{n-1}{n-k_n}\theta\\ &= \frac{n-1}{n-k_n}\left(\frac{n-k_n}{\xi}-1\right)\theta+\alpha^{-1}\frac{n-1}{\xi}\frac{z_0}{\sqrt{n}}\sqrt{\left(1+\frac{k_n-1}{n-k_n+1}u\right)\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}},\end{aligned}$$

where $\xi \sim \chi^2_{n-k}$, $z_0 \sim \mathcal{N}(0,1)$ and $u \sim \mathcal{F}(k-1, n-k+1, ns)$ with $s = (\boldsymbol{\mu}-r_f\mathbf{j}_k)^T\mathbf{R}_{\mathbf{l}}(\boldsymbol{\mu}-r_f\mathbf{j}_k)$ and $\mathbf{R}_{\mathbf{l}} = \boldsymbol{\Sigma}^{-1}-\boldsymbol{\Sigma}^{-1}\mathbf{l}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}/\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}$. Let us also note that ξ, u and z_0 are mutually independently distributed.

Since $\xi \sim \chi^2_{n-k}$, the application of Lemma 3 in Bodnar and Reiß (2016) leads us to

$$\frac{\xi}{n-k_n}-1\stackrel{a.s.}{\to}0 \qquad \text{and} \qquad \sqrt{n-k_n}\left(\frac{\xi}{n-k_n}-1\right)\stackrel{\mathcal{D}}{\to}\mathcal{N}(0,2) \tag{6}$$

for $k_n/n \to c \in (0,1)$ as $n \to \infty$.

We also have that

$$\sqrt{n-k_n}\frac{z_0}{\sqrt{n}}\stackrel{\mathcal{D}}{\to}\mathcal{N}(0,1-c). \tag{7}$$

Using the stochastic representation of a non-central $\mathcal{F}$-distributed random variable, i.e. $u = \frac{\zeta_1/(k_n-1)}{\zeta_2/(n-k_n+1)}$ with independent variables $\zeta_1 \sim \chi^2_{k_n-1,ns}$ and $\zeta_2 \sim \chi^2_{n-k_n+1}$, we obtain that

$$\begin{aligned}u-1-\frac{ns}{k_n-1} &= \frac{\zeta_1/(k_n-1)}{\zeta_2/(n-k_n+1)}-1-\frac{ns}{k_n-1}\\ &= \frac{n-k_n+1}{\zeta_2}\left[\left(\frac{\zeta_1}{k_n-1}-1-\frac{ns}{k_n-1}\right)-\left(1+\frac{ns}{k_n-1}\right)\left(\frac{\zeta_2}{n-k_n+1}-1\right)\right].\end{aligned}$$

From Lemma 3(a) in Bodnar and Reiß (2016) and using assumption (A1), we have that

$$\frac{\zeta_1}{k_n-1}-1-\frac{ns}{k_n-1}\stackrel{a.s.}{\to}0 \qquad \text{and} \qquad \frac{\zeta_2}{n-k_n+1}-1\stackrel{a.s.}{\to}0.$$

Consequently, it holds that

$$u-1-\frac{ns}{k_n-1}\stackrel{a.s.}{\to}0 \Rightarrow u\stackrel{a.s.}{\to}1+\frac{s}{c}.$$

Hence, we get

$$\sqrt{\left(1+\frac{k_n-1}{n-k_n+1}u\right)\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}}\stackrel{a.s.}{\to}\sqrt{\frac{1+s}{1-c}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}} \tag{8}$$

for $k_n/n \to c \in (0,1)$ as $n \to \infty$.

We also have that

$$\frac{n-1}{\xi} = \frac{n-1}{n-k}\frac{n-k}{\xi} \overset{a.s.}{\to} \frac{1}{1-c} \tag{9}$$

for $k_n/n \to c \in (0,1)$ as $n \to \infty$.

Taking into account (6), (7), (8) and (9), we get

$$\begin{aligned}\sqrt{n-k_n}\left(\hat{\theta} - \frac{n-1}{n-k_n}\theta\right) &= \frac{n-1}{\xi}\theta\sqrt{n-k_n}\left(1-\frac{\xi}{n-k_n}\right) \\ &\quad +\alpha^{-1}\frac{n-1}{\xi}\sqrt{\left(1+\frac{k_n-1}{n-k_n+1}u\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}}\sqrt{n-k_n}\frac{z_0}{\sqrt{n}} \\ &\overset{\mathcal{D}}{\to} \frac{1}{1-c}\theta z_1 + \alpha^{-1}\frac{1}{1-c}\sqrt{\frac{1+s}{1-c}\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}}z_2\end{aligned}$$

where $z_1 \sim \mathcal{N}(0,2)$ and $z_2 \sim \mathcal{N}(0,1-c)$ and they are independently distributed.

Finally, the application of the properties of normal random variables leads to

$$\sqrt{n-k_n}\sigma^{-1}\left(\hat{\theta} - \frac{n-1}{n-k_n}\theta\right) \overset{\mathcal{D}}{\to} \mathcal{N}(0,1),$$

where

$$\sigma^2 = \frac{2}{(1-c)^2}\theta^2 + \frac{1}{(1-c)^2}\alpha^{-2}\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l} + \frac{1}{(1-c)^2}\alpha^{-2}s\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}.$$

Let us note that

$$\begin{aligned}s &= (\boldsymbol{\mu} - r_f\mathbf{j}_k)^T\mathbf{R_l}(\boldsymbol{\mu} - r_f\mathbf{j}_k) \\ &= (\boldsymbol{\mu} - r_f\mathbf{j}_k)^T\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{j}_k) - (\boldsymbol{\mu} - r_f\mathbf{j}_k)^T\mathbf{\Sigma}^{-1}\mathbf{l}\mathbf{l}^T\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{j}_k)/\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\end{aligned}$$

and, therefore, we get that

$$\alpha^{-2}s\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l} = \alpha^{-2}\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}(\boldsymbol{\mu} - r_f\mathbf{j}_k)^T\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{j}_k) - \theta^2.$$

Hence, we obtain that

$$\sigma^2 = \frac{\alpha^{-2}}{(1-c)^2}\left[\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l} + (\alpha\theta)^2 + \mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}(\boldsymbol{\mu} - r_f\mathbf{j}_k)^T\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{j}_k)\right].$$

The statement of the theorem is proved.

□

Proof. From the proof of Proposition 1 of Bodnar and Schmid (2009), we know that the conditional distribution of T is given by

$$\begin{aligned} T|\hat{s}=y &\sim t_{n-k_n,\upsilon(y)} \\ \frac{n(n-k_n+1)}{(k_n-1)(n-1)}\hat{s} &\sim \mathcal{F}_{k_n-1,n-k_n+1,ns} \end{aligned}$$

with $\upsilon(y) = \frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{n}{n-1}y\right)}}$.

Using the stochastic representation of a non-central t-distribution, we obtain

$$\begin{aligned} T|\hat{s}=y &\overset{d}{=} \frac{\frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{n}{n-1}y\right)}}+z_0}{\sqrt{\frac{\xi}{n-k_n}}} \\ &= \sqrt{\frac{n-k_n}{\xi}}\left(\frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k_n-1}{n-k_n+1}u\right)}}+z_0\right) \end{aligned}$$

where $z_0 \sim \mathcal{N}(0,1)$, $\xi \sim \chi^2_{n-k_n}$ and $u = \frac{n(n-k_n+1)}{(k_n-1)(n-1)}\hat{s} \sim \mathcal{F}_{k_n-1,n-k_n+1,ns}$; moreover, z_0, ξ and u are independent.

We then have

$$\begin{aligned} &T - \frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)\right)}} \\ =& \sqrt{\frac{n-k_n}{\xi}}\left(\frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k_n-1}{n-k_n+1}u\right)}}+z_0\right) - \frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)\right)}} \\ =& \sqrt{\frac{n-k_n}{\xi}}z_0 + \frac{\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k_n-1}{n-k_n+1}u\right)}} \\ &\times\left(\sqrt{n}\left(\sqrt{\frac{n-k_n}{\xi}}-1\right)+\sqrt{n}\left(1-\frac{\sqrt{1+\frac{k_n-1}{n-k_n+1}u}}{\sqrt{\left(1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)\right)}}\right)\right). \end{aligned}$$

Putting the third term on a common denominator, we get

$$1-\frac{\sqrt{1+\frac{k_n-1}{n-k_n+1}u}}{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}} = \frac{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}-\sqrt{1+\frac{k_n-1}{n-k_n+1}u}}{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}}.$$

Multiplying the numerator of the last expression by its conjugate, we obtain

$$1-\frac{\sqrt{1+\frac{k_n-1}{n-k_n+1}u}}{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}} = \frac{\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s-u\right)}{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}+\sqrt{1+\frac{k_n-1}{n-k_n+1}u}} \times\frac{1}{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}}.$$

Applying results from the proof of Theorem 1, we have that

$$u \overset{a.s.}{\to} 1+\frac{s}{c} \Rightarrow \sqrt{1+\frac{k_n-1}{n-k_n+1}u} \overset{a.s.}{\to} \sqrt{\frac{1+s}{1-c}} \tag{10}$$

and

$$\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)} \overset{a.s.}{\to} \sqrt{\frac{1+s}{1-c}}. \tag{11}$$

Using (10) and (11), the denominator becomes

$$\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}+\sqrt{1+\frac{k_n-1}{n-k_n+1}u}\frac{1}{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}} \overset{a.s.}{\to} 2\frac{1+s}{1-c} \tag{12}$$

for $k_n/n \to c \in (0,1)$ as $n \to \infty$.

Using the stochastic representation of a non-central $\mathcal{F}$ distribution

$$u \overset{d}{=} \frac{\eta_1/(k_n-1)}{\eta_2/(n-k_n+1)}$$

with independent random variables $\eta_1 \sim \chi^2_{k_n-1,ns}$ and $\eta_2 \sim \chi^2_{n-k_n+1}$, we have that

$$\begin{aligned} 1+\frac{ns}{k_n-1}-u &\overset{d}{=} 1+\frac{ns}{k_n-1}-\frac{\eta_1/(k_n-1)}{\eta_2/(n-k_n+1)} \\ &= \frac{1}{\eta_2/(n-k_n+1)}\left[\left(\frac{\eta_2}{n-k_n+1}-1\right)\left(1+\frac{ns}{k_n-1}\right)\right. \\ &\quad \left.-\left(\frac{\eta_1}{k_n-1}-1-\frac{ns}{k_n-1}\right)\right]. \end{aligned}$$

Applying Lemma 3 in Bodnar and Reiß (2016), we obtain

$$\frac{\eta_2}{n-k_n+1} \overset{a.s.}{\to} 1 \qquad \text{and} \qquad \frac{\eta_1}{k_n-1}-1-\frac{n}{k_n-1}s \overset{a.s.}{\to} 0;$$

moreover, it holds that

$$\sqrt{n}\left(\frac{\eta_2}{n-k_n+1}-1\right)\overset{\mathcal{D}}{\to}\mathcal{N}\left(0,\frac{2}{1-c}\right),$$
$$\sqrt{n}\left(\frac{\eta_1}{k_n-1}-1-\frac{ns}{k_n-1}\right)\overset{\mathcal{D}}{\to}\mathcal{N}\left(0,\frac{2}{c}\left(1+2\frac{s}{c}\right)\right),$$

for $k_n/n \to c \in (0,1)$ as $n \to \infty$.

The application of Slutsky's lemma leads to

$$\sqrt{n}\left(1+\frac{ns}{k_n-1}-u\right)\overset{\mathcal{D}}{\to}\mathcal{N}\left(0,\frac{2}{c}\left(1+2\frac{s}{c}\right)+\frac{2}{1-c}\left(1+\frac{s}{c}\right)^2\right). \tag{13}$$

Hence, since $\xi \sim \chi^2_{n-k_n}$ and the usage of (6) we have that

$$\sqrt{n}\left(\sqrt{\frac{n-k_n}{\xi}}-1\right)=\frac{\sqrt{n}\left(1-\frac{\xi}{n-k}\right)\frac{n-k}{\xi}}{\sqrt{\frac{n-k}{\xi}}+1}\overset{\mathcal{D}}{\to}\frac{1}{2}\tilde{z}_1 \tag{14}$$

where $\tilde{z}_1 \sim \mathcal{N}(0, \frac{2}{1-c})$.

Using (12) and (13), we obtain

$$\sqrt{n}\left(1-\frac{\sqrt{1+\frac{k_n-1}{n-k_n+1}u}}{\sqrt{\left(1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)\right)}}\right)\overset{\mathcal{D}}{\to}\frac{\frac{c}{1-c}}{2\frac{1+s}{1-c}}\tilde{z}_2 \tag{15}$$

where $\tilde{z}_2 \sim \mathcal{N}(0,\sigma_0^2)$ with $\sigma_0^2 = \frac{2}{c}\left(1+2\frac{s}{c}\right)+\frac{2}{1-c}\left(1+\frac{s}{c}\right)^2$.

Putting everything together and taking into account (8), (14) and (15) we obtain

$$T-\frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)\right)}}\overset{\mathcal{D}}{\to}\tilde{z}_0+\frac{\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\frac{1+s}{1-c}}}\left(\frac{1}{2}\tilde{z}_1+\frac{\frac{c}{1-c}}{2\frac{1+s}{1-c}}\tilde{z}_2\right)$$

where $\tilde{z}_0 \sim \mathcal{N}(0,1)$ and $\tilde{z}_1$ and $\tilde{z}_2$ are defined in (14) and (15), respectively. Moreover, $\tilde{z}_0$, $\tilde{z}_1$ and $\tilde{z}_2$ are independent.

Finally, the application of the properties of normal random variables leads to

$$\sigma_T^{-1}\left(T-\frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)\right)}}\right)\overset{\mathcal{D}}{\to}\mathcal{N}(0,1)$$

with

$$
\begin{aligned}
\sigma_T^2 &= 1 + \frac{(\alpha\rho)^2}{\mathbf{l}^T \mathbf{\Sigma}^{-1}\mathbf{l}(1+s)}(1-c)\left(\frac{1}{2(1-c)} + \frac{c^2}{4(1+s)^2}\sigma_u^2\right) \\
&= 1 + \frac{(\alpha\rho)^2}{\mathbf{l}^T \mathbf{\Sigma}^{-1}\mathbf{l}(1+s)}\left(\frac{1}{2} + \frac{s^2+2s+c}{2(1+s)^2}\right).
\end{aligned}
$$

The statement of Theorem 2(b) follows by setting $\rho = 0$ under the null hypothesis. The theorem is proved. □

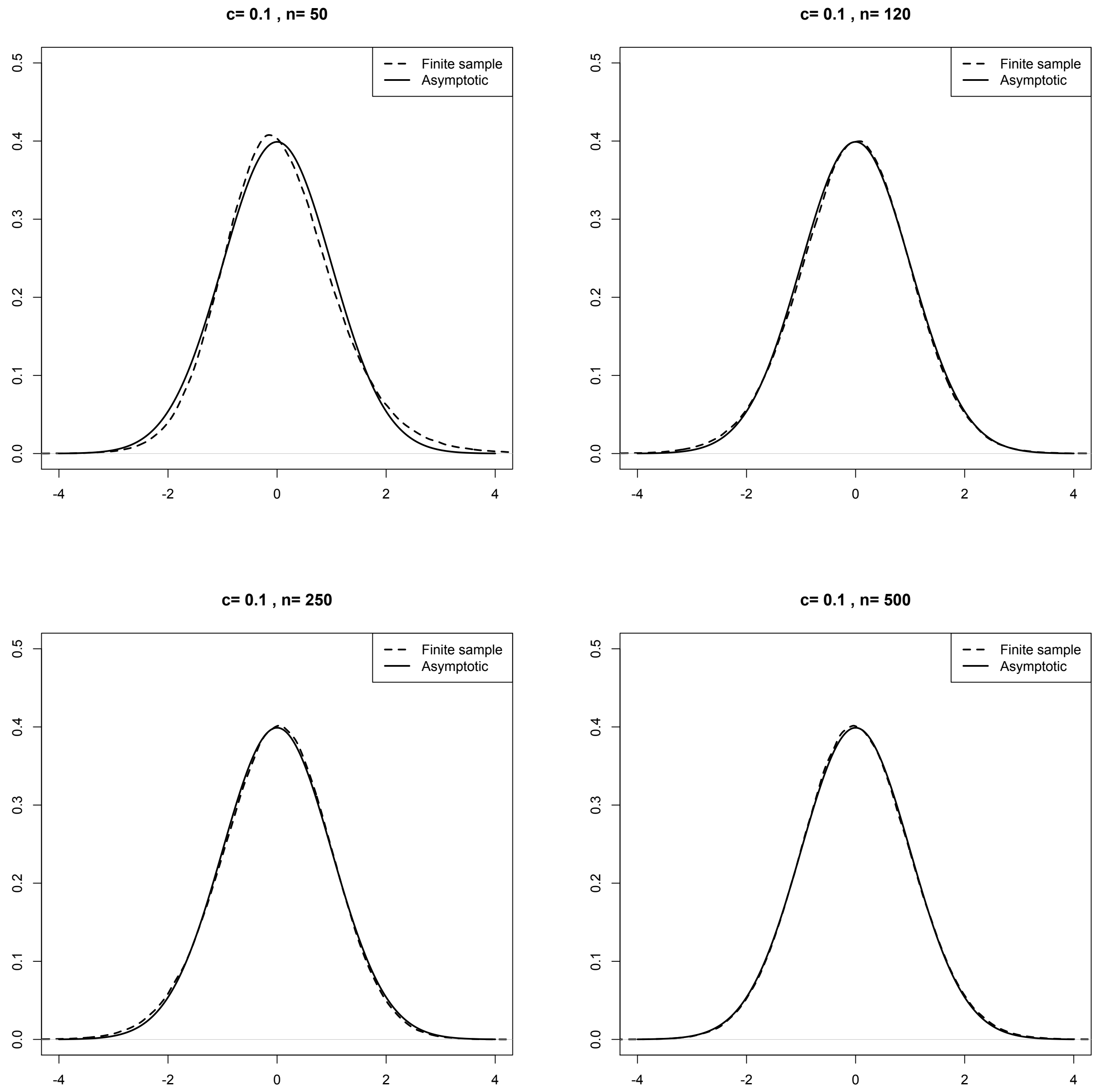


Figure 3: Asymptotic distribution and the kernel density estimator of the finite sample distribution of standardized $\hat{\theta}$ for $c = 0.1$.

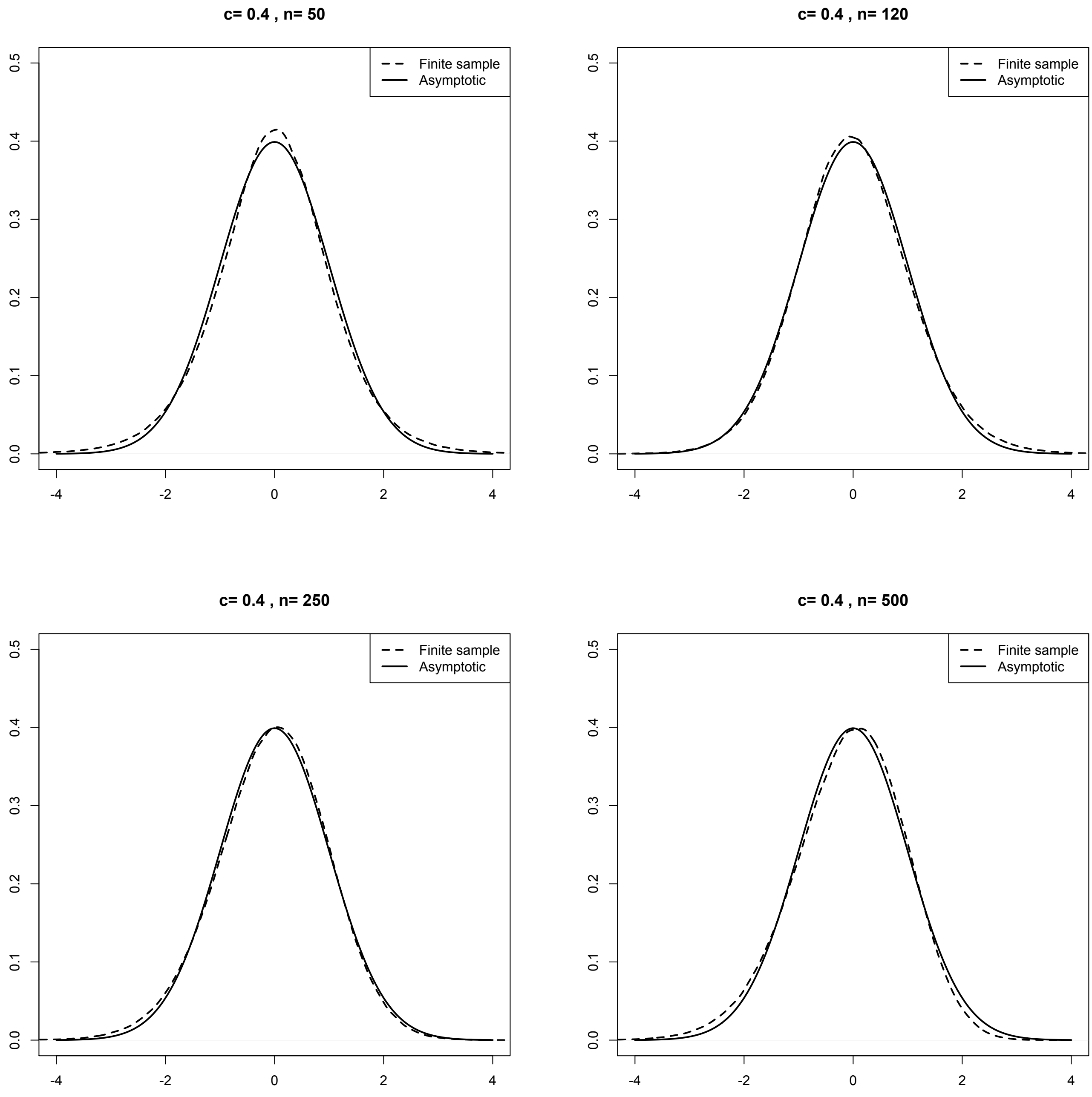


Figure 4: Asymptotic distribution and the kernel density estimator of the finite sample distribution of standardized $\hat{\theta}$ for $c = 0.4$.

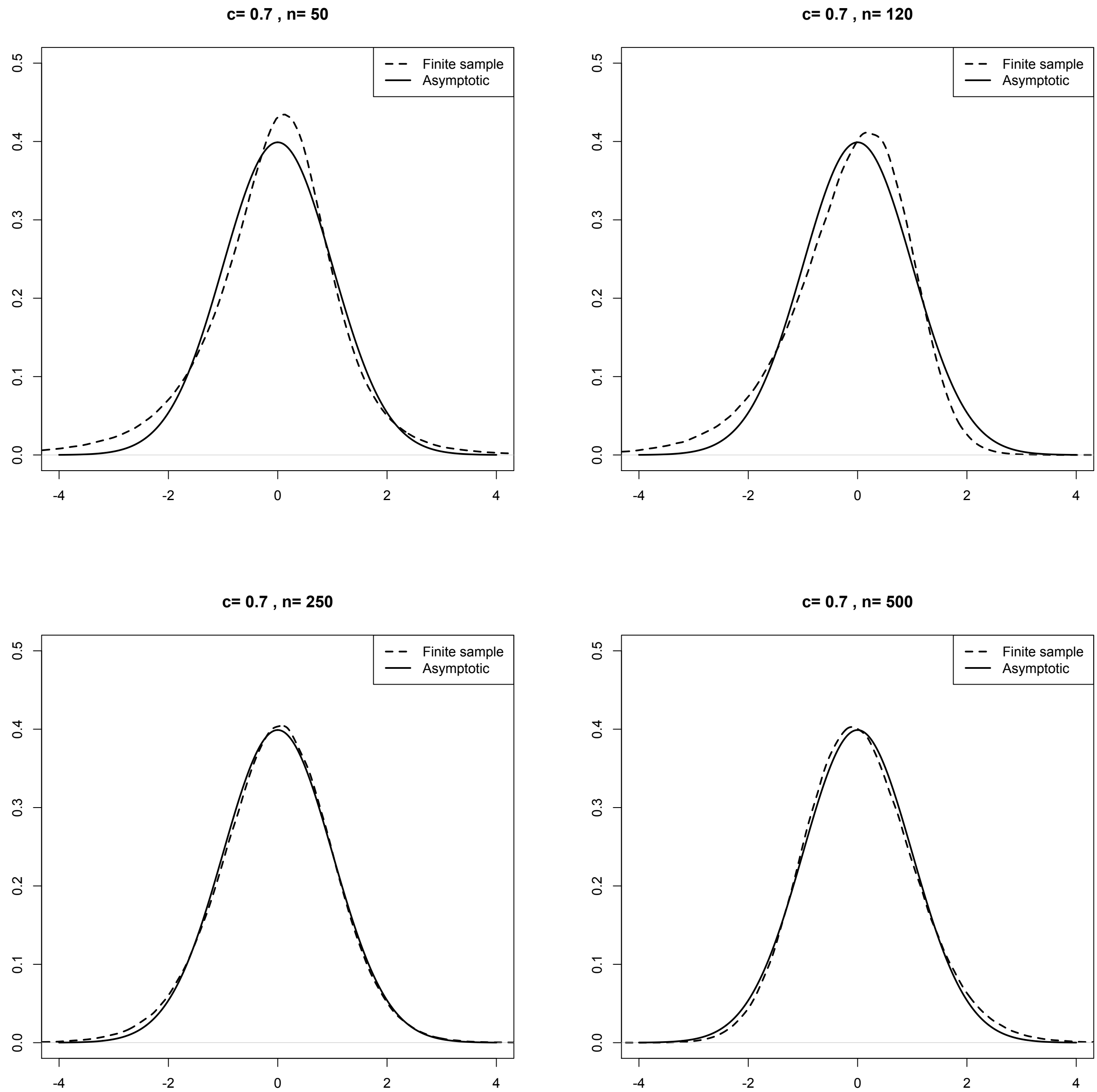


Figure 5: Asymptotic distribution and the kernel density estimator of the finite sample distribution of standardized $\hat{\theta}$ for $c = 0.7$.

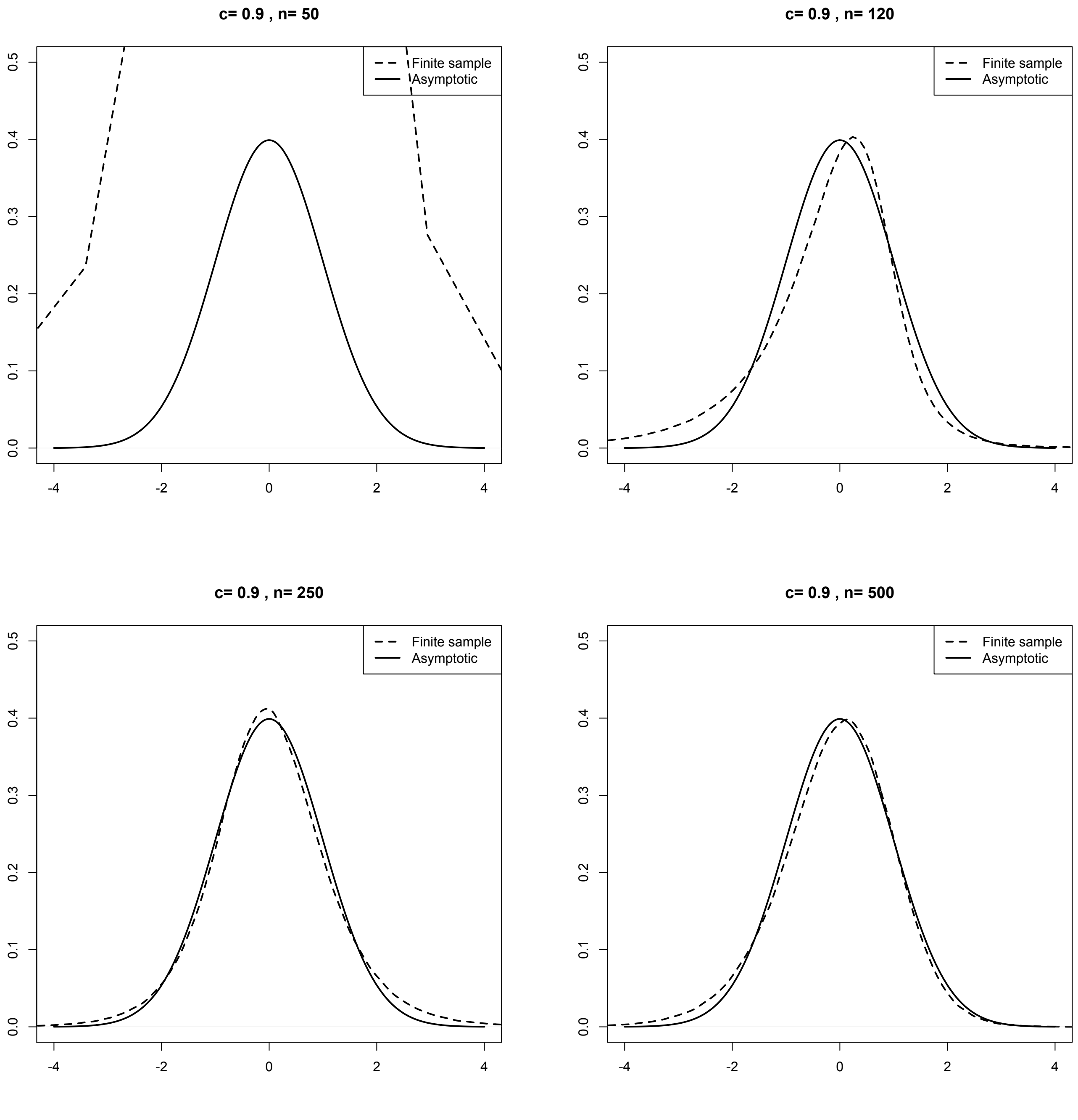


Figure 6: Asymptotic distribution and the kernel density estimator of the finite sample distribution of standardized $\hat{\theta}$ for $c = 0.9$.

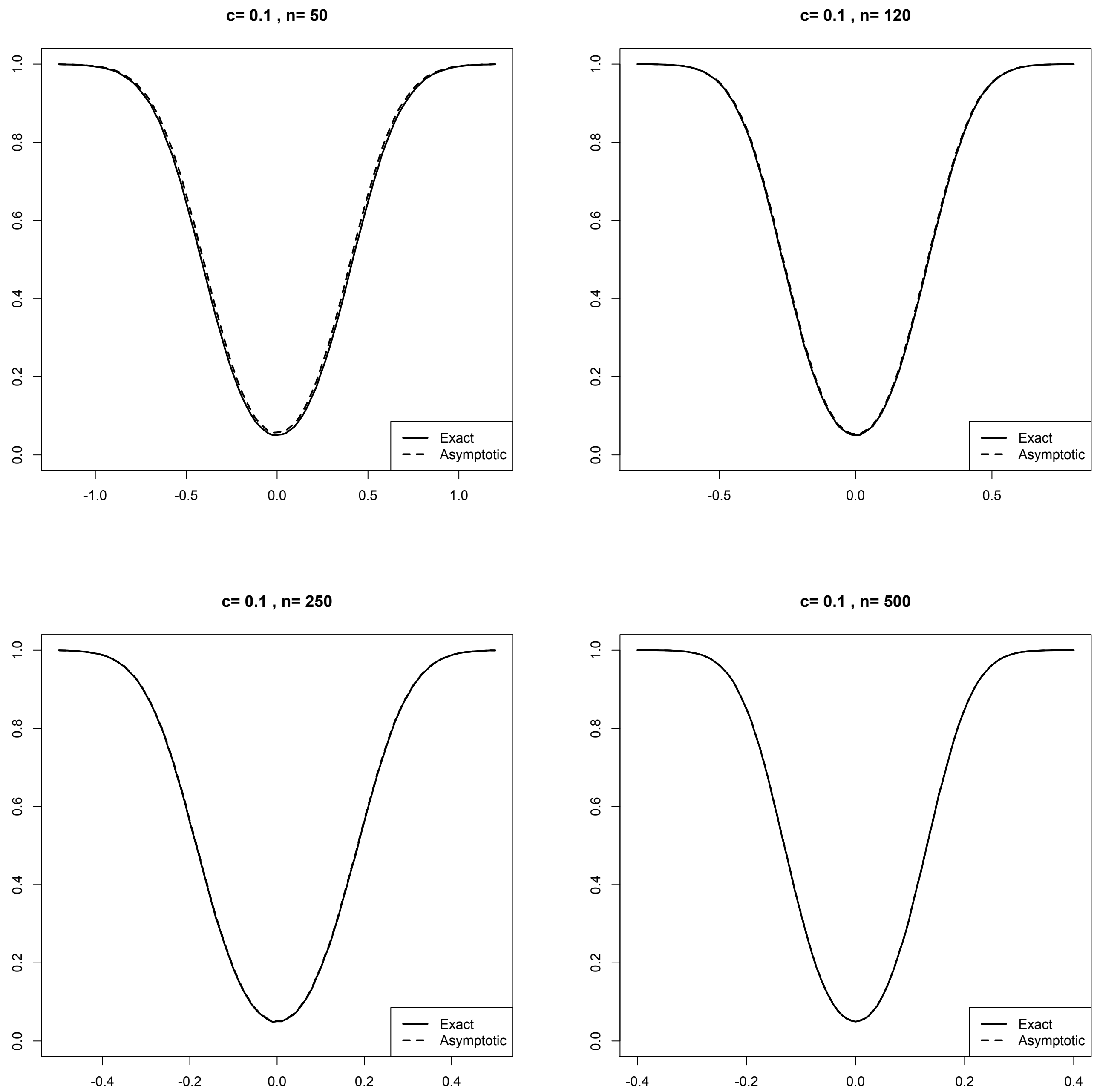


Figure 7: Powers of the exact test and of the high-dimensional asymptotic test as a function of λ based on the statistic T for $c = 0.1$ with $s = 1$ and $\psi = 5\%$.

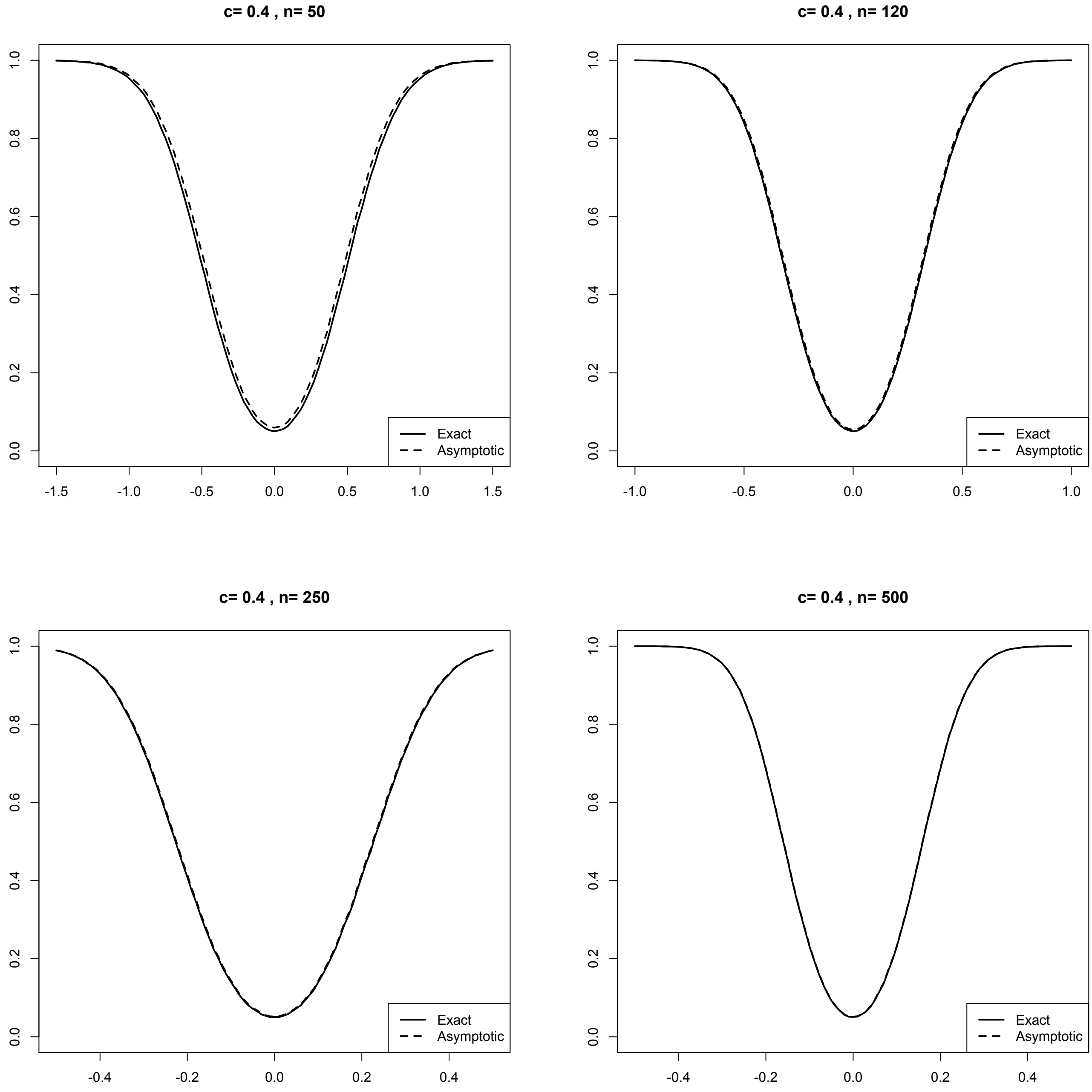


Figure 8: Powers of the exact test and of the high-dimensional asymptotic test as a function of λ based on the statistic T for $c = 0.4$ with $s = 1$ and $\psi = 5\%$.

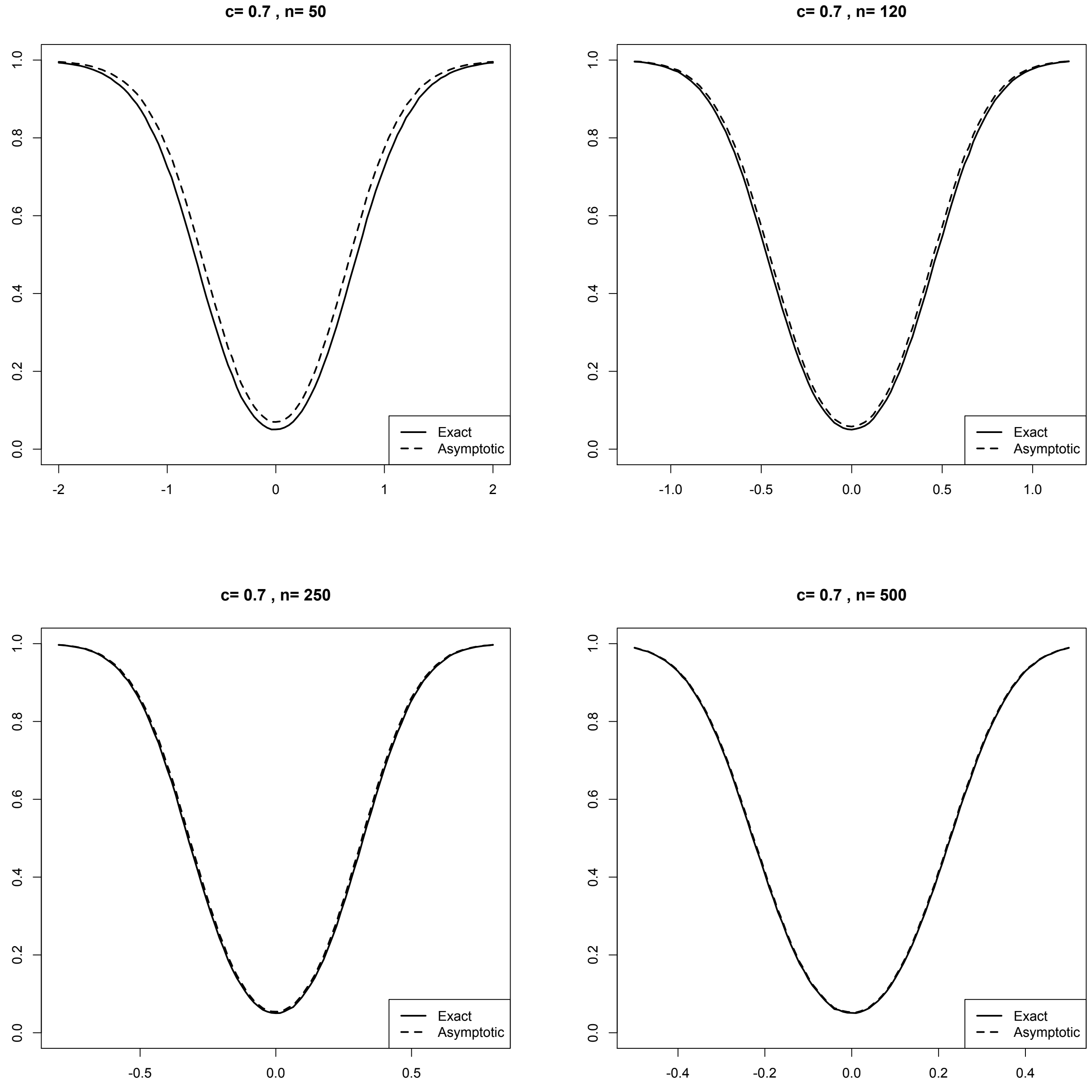


Figure 9: Powers of the exact test and of the high-dimensional asymptotic test as a function of λ based on the statistic T for $c = 0.7$ with $s = 1$ and $\psi = 5\%$.

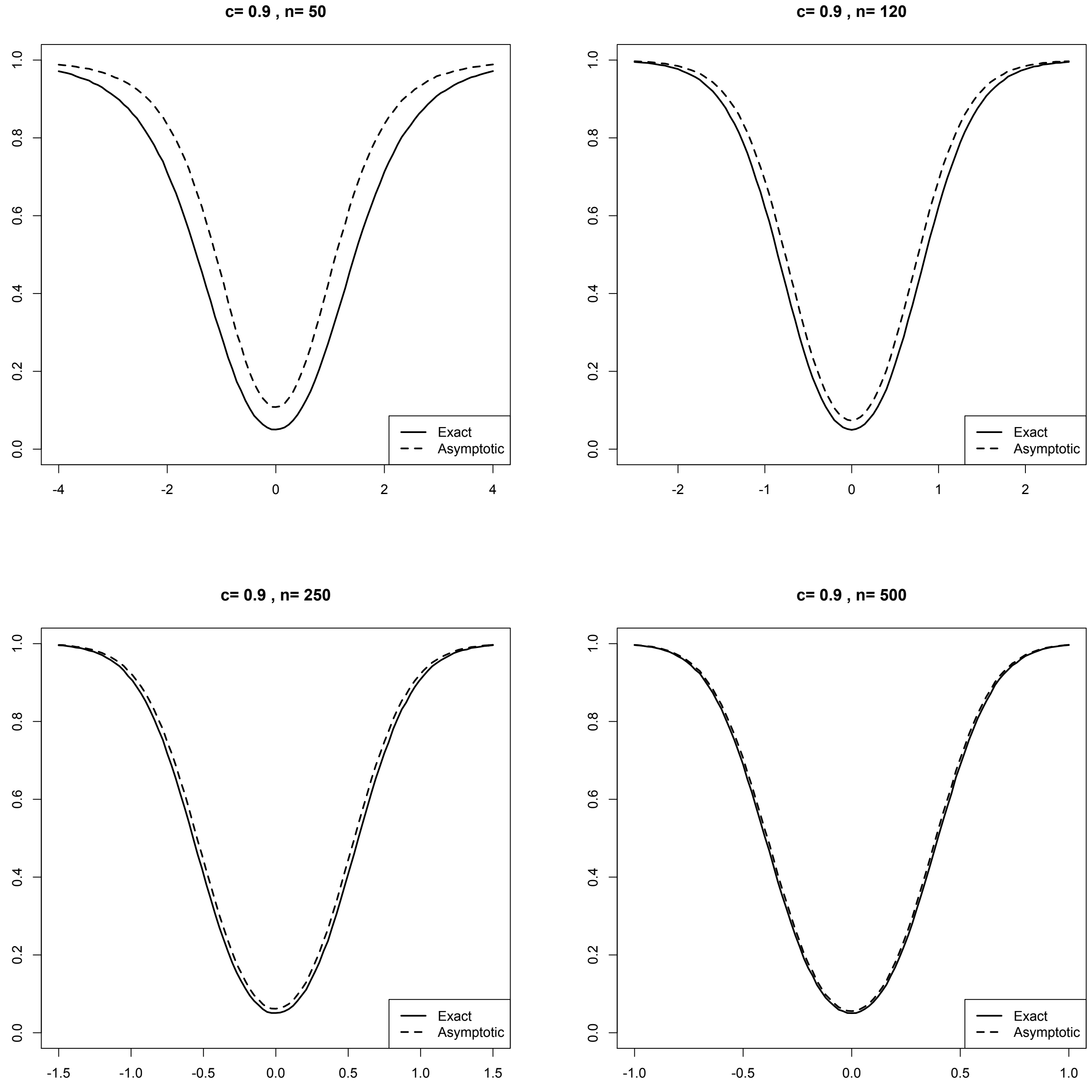
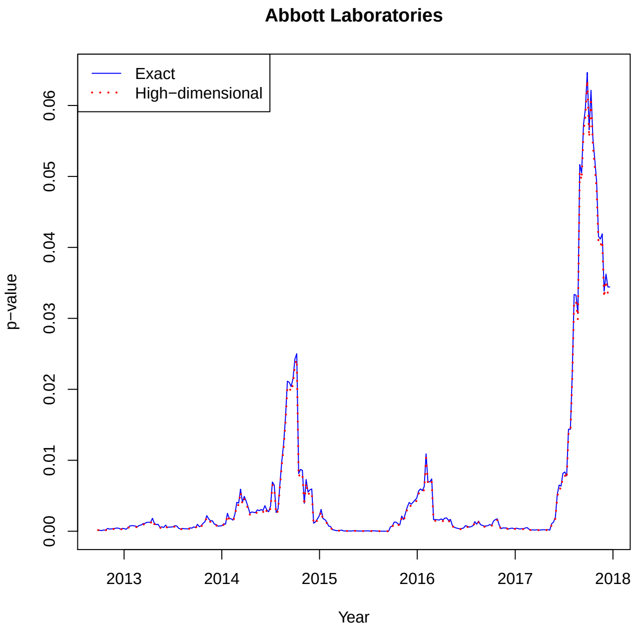
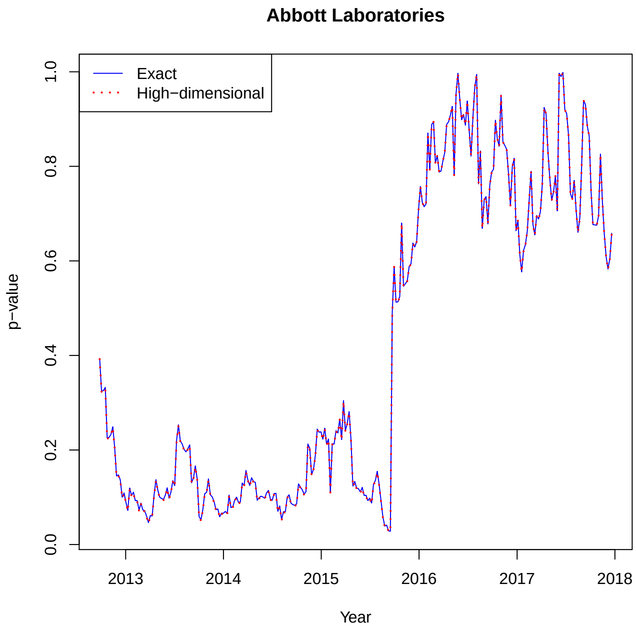


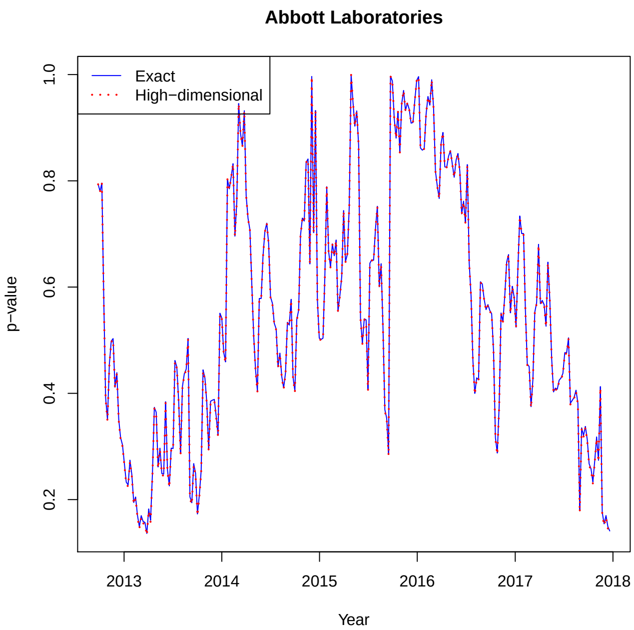
Figure 10: Powers of the exact test and of the high-dimensional asymptotic test as a function of λ based on the statistic T for $c = 0.9$ with $s = 1$ and $\psi = 5\%$.



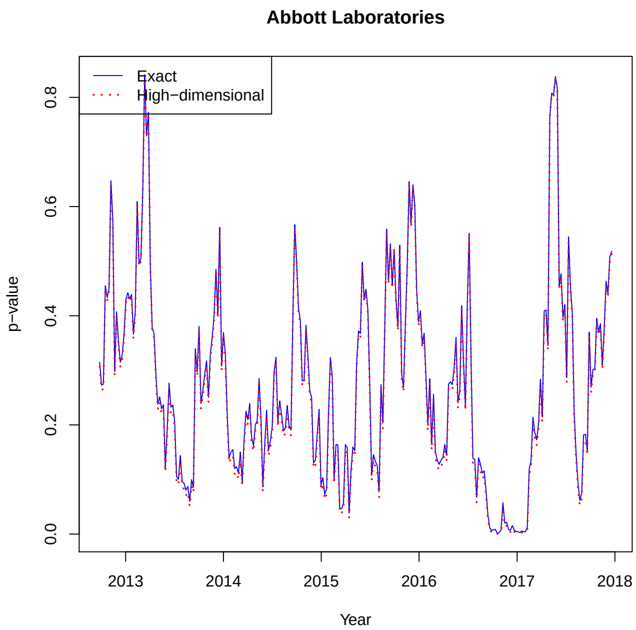
(a) $k = 30$



(b) $k = 120$

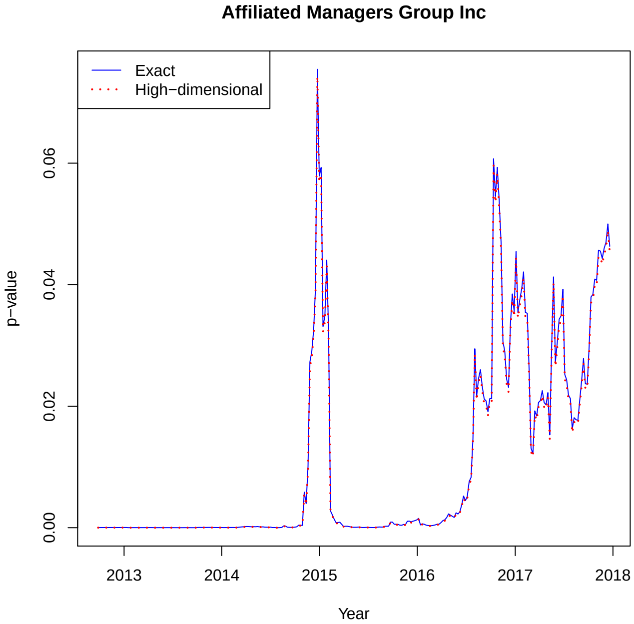


(c) $k = 210$

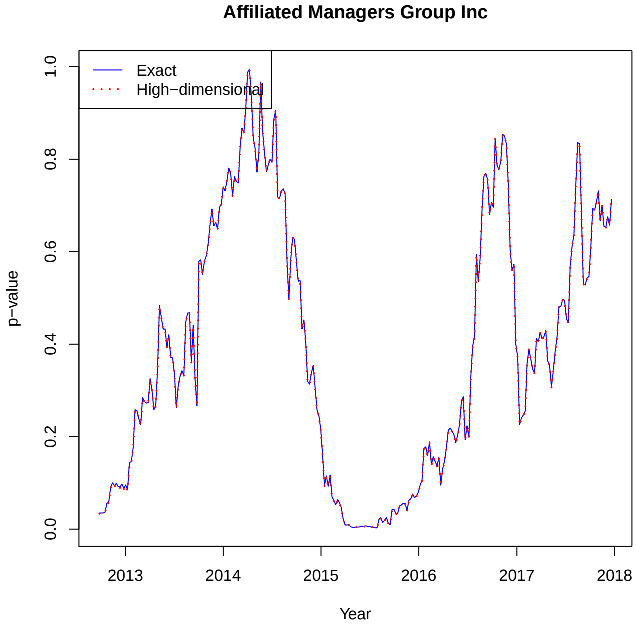


(d) $k = 270$

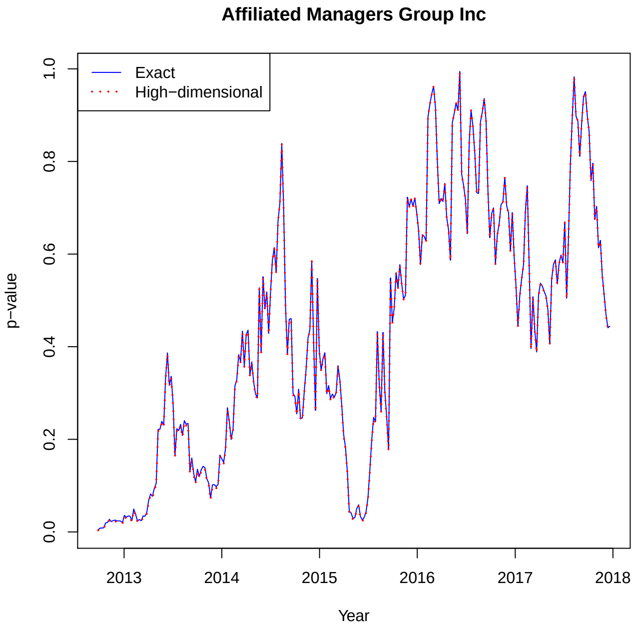
Figure 11: p-values of the exact and the asymptotic tests on the tangency portfolio weight of Abbott Laboratories.



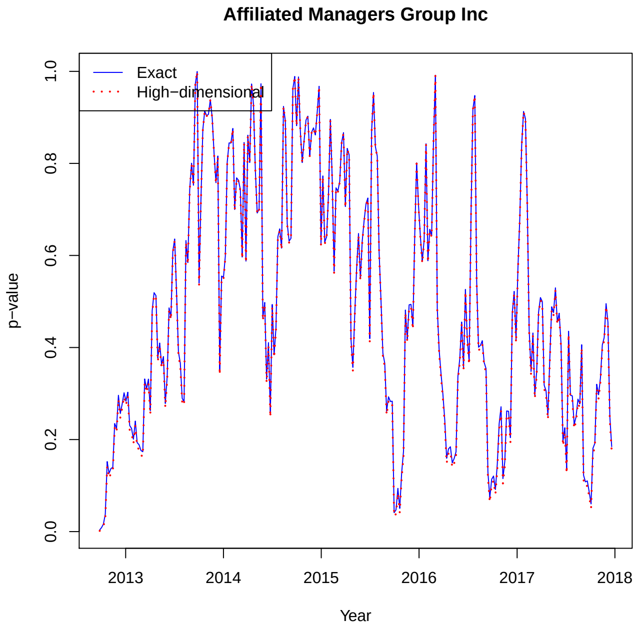
(a) $k = 30$



(b) $k = 120$

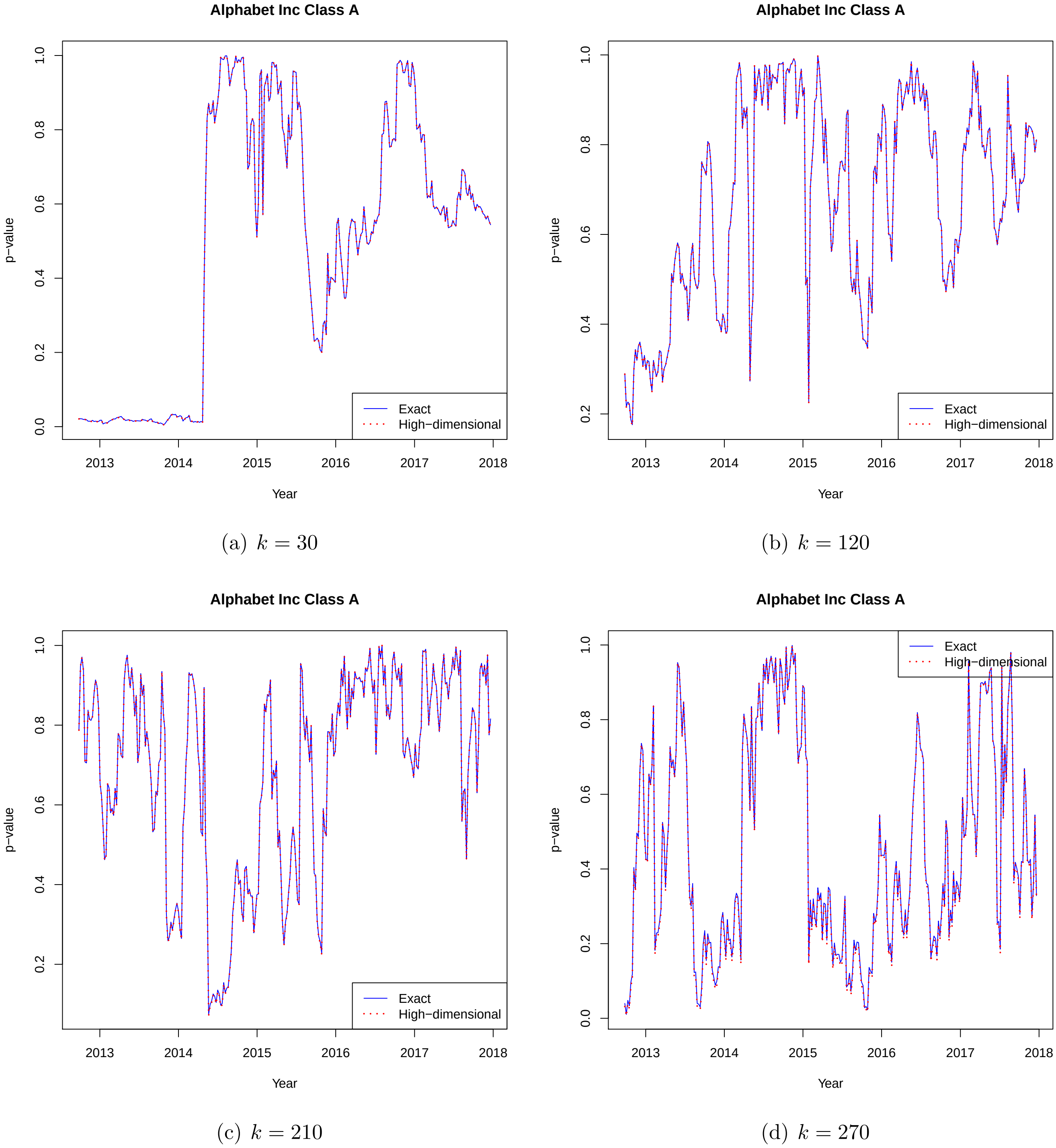


(c) $k = 210$



(d) $k = 270$

Figure 12: p-values of the exact and the asymptotic tests on the tangency portfolio weight of Affiliated Managers Group Inc.



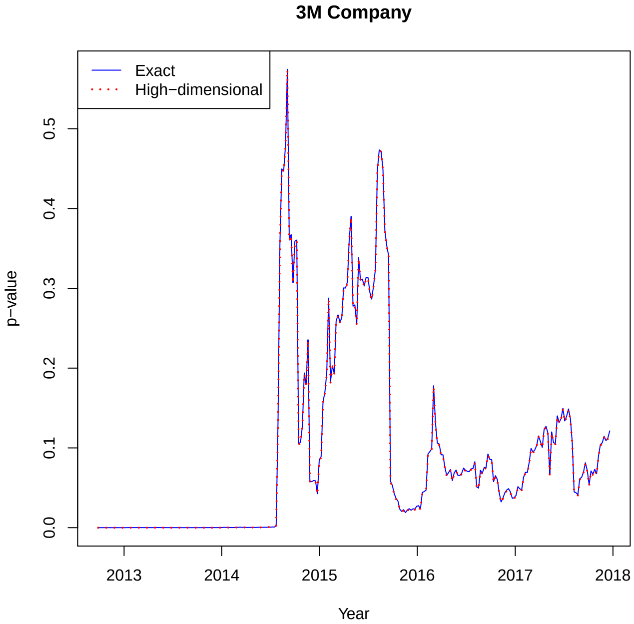
(a) $k = 30$

(b) $k = 120$

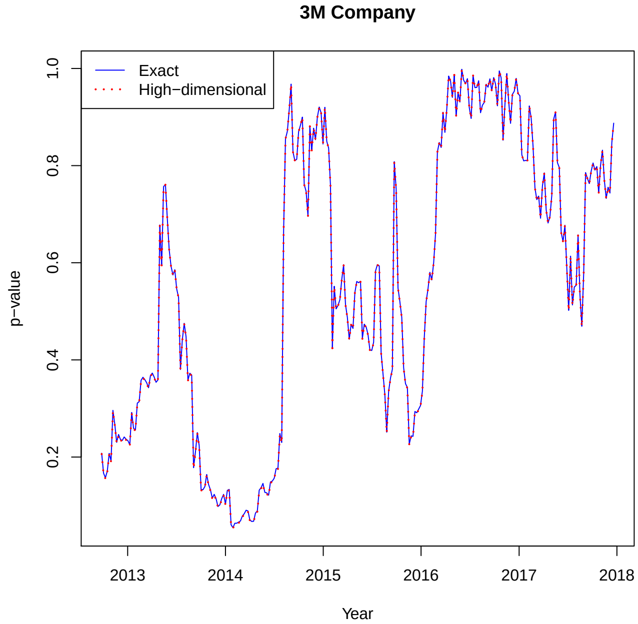
(c) $k = 210$

(d) $k = 270$

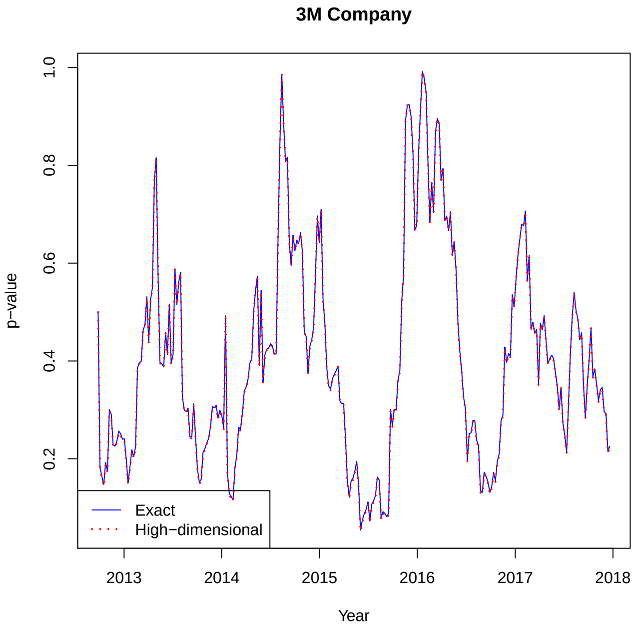
Figure 13: p-values of the exact and the asymptotic tests on the tangency portfolio weight for Alphabet Inc Class A.



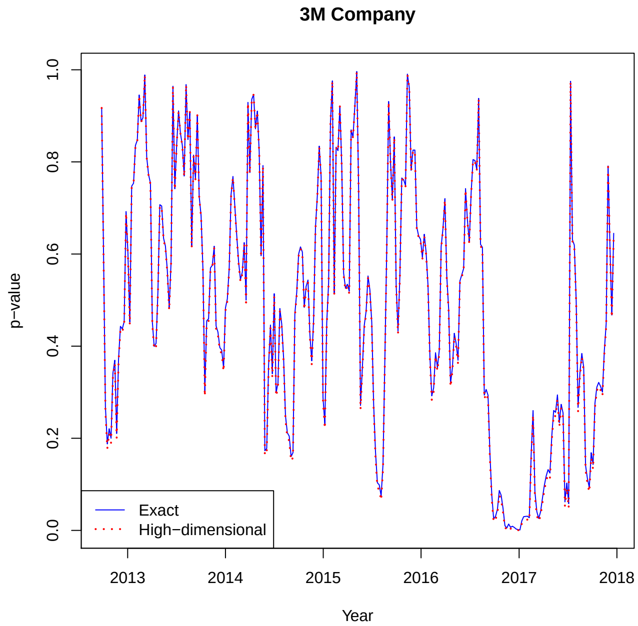
(a) $k = 30$



(b) $k = 120$



(c) $k = 210$



(d) $k = 270$

Figure 14: p-values of the exact and the asymptotic tests on the tangency portfolio weight of 3M Company.