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Is Miss Piggy Going in for Final Landing?
A Statistical Analysis of the Game *Pass the Pigs*®

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Pigs in Space:
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Abstract

The game Pass the Pigs® offers students in statistics to practice a lot of statistical concepts and methods in a coherent way. In this paper a probability model is derived based on assumptions of the data generating process. A previously suggested strategy of maximizing the expected score in each round is adapted to the model. Adopting this strategy requires estimating the score of accumulated points, where to stop rolling the pigs and pass them on to the next player, as a function of unknown parameters in the model. Point and interval estimators for the mentioned accumulated score are derived using the maximum likelihood method as well as the Delta method and Bootstrap methods. Data from an experiment is used to testing model assumptions and to get point and interval estimates.

JEL Classification: A22, A23

Keywords: Pigs, Game, Strategy, Delta Method, Bootstrap

1. Introduction

In the game Pass the Pigs® created by David Moffat players are to roll two small plastic pigs that land in different configurations. The player earns or loses points depending on the way the pigs land. A player's turn, or round, ends, and the pigs are passed on to the next player, when either the player gets certain configurations or when the player himself or herself decides to stop. The probabilities of the various configurations are unknown due to the shape of the pigs. In order for the player to make strategic decisions on when to stop, knowledge of these probabilities are required.

In this paper data is generated from an experiment of 5000 rolls of the two plastic pigs and used in a proposed model to make inference on the score where to stop rolling the pigs in a given round in order to maximize the expected score.

There is a literature on games as motivating tools for probability and statistics learning. Wilson et al. (2009) discusses a probability project using the game Yahtzee in a probability course to help students transition from theory to application. In order to illustrate inferences on proportions and discrete probability distributions Stephenson et al. (2009) make use of the dice-based golf game GOLO. Bohan and Schultz (1996) and Feldman and Morgan (2003) suggest the game HOG as a pedagogical tool for illustrating areas including analytical concepts.

Two research articles discuss the game *Pass the Pigs*®. Kern (2006) adopts a Bayesian approach to make inference about the configuration probabilities and predictive posterior simulations are used to compare different game strategies. This presentation is intended for students of Bayesian inference at an advanced undergraduate or first year graduate level. Another approach is taken by Gorman (2012). He describes how a classroom exercise can be structured to train business students in various topics such as data collection and preparation, model framing, probability, optimization, expected value decision making and simulation.

There are three more articles written about a similar game *Pig*, where the plastic pigs are replaced by dice and the probabilities of the outcomes are, therefore, known. In all three articles the pedagogical value of the game is at focus. Shi (2000) helps teachers to show students how a winning strategy might be developed based on mathematical thinking, while Neller and Presser (2004) use the algorithm value iteration to solve for an optimal winning strategy. Neller at al. (2006) describe the game's historical uses in mathematics and discuss ways in which the game can be instructional in teaching concepts in computer science courses as well as in introductory artificial intelligence, networking, and scientific visualization courses.

Thus, five research articles use the game *Pass the Pigs*® and the related game *Pig* for illustrating topics in probability and statistics. Is there really a need for adding yet another article to this collection? Yes, it is. This paper focus foremost on inferential aspects related to the game, while the other articles have another focus, except for Kern (2006). However, although he also concentrates on inference, his approach is Bayesian while this paper makes use of classical inference methods.

The analysis in this paper is intended for advanced undergraduate or first year graduate students taking a course in Statistical theory. The problem might serve as a comprehensive home assignment at the end of the course to tie together all the topics covered. Another approach is to split up the problem into several parts, given as smaller home assignments as concepts and methods are introduced during the course.

The rest of the paper is organized as follows: Section 2 contains a brief description of the game. The data is presented in section 3. A model based on certain assumptions of the data generating process is developed in section 4 and the strategy of maximizing the expected score in a given round adapted to the model is also accounted for here. In section 5, using this strategy, point and interval estimators for the score of accumulated points, where to stop rolling the pigs and pass them on to the next player, are derived using the maximum likelihood method as well as the Delta method and Bootstrap methods. This section also includes a proposed test for model assumptions. Section 6 contains estimation and testing results, while the paper ends with a discussion in section 7.

2. Description of the Game

Two or more people can participate in the game Pass the Pigs®, where the first player to reach 100 points win the game. One player is randomly chosen to go first. Play then goes on in a certain order on a turn-by-turn basis. Points earned by a player in a particular round is added to previous rounds. On a player's turn he or she rolls both the pigs together on a smooth surface and note the score, where the score is determined according to the configuration, the way the pigs land. If the score is positive the player may continue to roll the pair of pigs trying to get more points to add to the score. This accumulation of points in a player's round is possible if the previous roll results in a positive-scoring configuration. The round ends when the player passes the pigs onto the next player. This happens with the occurrence of one of the following three events:

- The roll is positive-scoring and the player decides to stop rolling and score the accumulated points earned in this round.
- The roll results in a configuration where the pigs land on opposite sides. In this case all the accumulated points in this round are lost and the score from this round is therefore zero.
- The roll results in an event where the pigs land and touch each other at rest in any position. All points earned in the game so far is lost, that is, all points in this particular round as well as all the accumulated points in previous rounds.

In Table 1 below the seven possible positions for one pig, in a roll with both pigs, are described and named. The names are the same as given in Kern (2006), except for the seventh position, which is included here.

Table 1. Possible positions for one pig in a roll with both pigs.

Position	Name	Description
1	Dot up	Pig lies on its left side
2	Dot down	Pig lies on its right side
3	Trotter	Pig lands on all four feet
4	Razorback	Pig lies on its back
5	Snouter	Pig balances on its snout and two front-feet
6	Leaning Jowler	Pig balances on a front foot, ear and snout
7	Touch	Pig touches the other pig

The Dot up and Dot down labels come from the fact that each pig is marked with a black dot on the right side of the body. Note that only one position can occur at the time, since position 7 is superior to the other positions. For example, if one pig lies on the left side, while touching the other pig, the position is not 1, rather 7.

Different combinations of the positions given above for the two pigs rolled together result in different points given to or taken from the player. In Table 2 gain in points of one more roll for different pig positions is shown, where T is the accumulated score in the ongoing round, while S is the accumulated score from previous rounds.

Table 2. Gain in scores of one more roll for different pig positions.

	Position of Pig 2						
Position of Pig 1	1	2	3	4	5	6	7
1	1	-T	5	5	10	15	
2	-T	1	5	5	10	15	
3	5	5	20	10	15	20	
4	5	5	10	20	15	20	
5	10	10	15	15	40	25	
6	15	15	20	20	25	60	
7							-(S+T)

Thus, for all but three combinations the gain is positive, where the highest gain is attained for the pair of combination $(6, 6)$ when both pigs land on their jowls, supported by a snout, an ear and a front foot.

3. The Data

The data was generated by one person who rolled the pair of pigs 5000 times on a wooden table with a smooth surface.

Table 3. Frequencies of pig positions based on the experiment of rolling the pair of pigs 5000 times.

	Position of Pig 2						
Position of Pig 1	1	2	3	4	5	6	7
1	445	527	147	318	33	8	
2	519	581	164	420	47	5	
3	136	156	49	130	8	1	
4	325	381	95	262	37	4	
5	59	45	16	34	2	1	
6	9	9	5	3	0	0	
7							19

The pigs were shaken three times before dropping them onto the table from a height of approximately 20 centimeters from the surface of the table. After each roll the positions of the pigs were recorded. Following Kern (2006) one pig was randomly selected and marked, here with a blue dot, in order to allow for any dissimilarity between the pigs. Table 3 on the previous page shows frequencies of pig positions based on the 5000 rolls, where Pig 1 refers to the pig with a blue dot, and Pig 2 therefore is the unmarked pig.

The data will be used to estimate the probabilities of different combinations of pig positions yielding different scores with the ultimate purpose of making inference on the accumulated score, where to stop rolling the pigs in order to maximize the expected score in a given round.

4. The Model

4.1 A Joint probability Model

The pair of pigs, denoted by Pig 1 and Pig 2, are rolled and for each pig one out of the seven possible outcomes described in the previous section is observed. To this random experiment we associate the joint discrete random variables X_1 and X_2, indicating the outcome of each pig. To be more specific, X_1 takes the value i if the outcome of the first pig is i and, in a similar way, X_2 takes the value j if the outcome of the second pig is j. The simultaneous probabilities are denoted by p_{ij}, while we denote the marginal probabilities by $p_{i.}$ and $p_{.j}$. From the description of the rules of the game it is evident that $p_{i7} = p_{7j} = 0$ for $i = 1, \dots, 6$ and $j = 1, \dots, 6$, respectively. Moreover, $p_{7.} = p_{.7} = p_7$ and $p_{77} = p_7$. Thus, we get a joint probability distribution for X_1 and X_2 as

$$P(X_1 = i, X_2 = j) = \begin{cases} p_{ij}, & i = 1, \dots, 6, j = 1, \dots, 6, \\ p_i, & i = j = 7. \end{cases} \quad (1)$$

The marginal distributions are given by

$$P(X_1 = i) = \begin{cases} p_{i.}, & i = 1, \dots, 6, \\ p_i, & i = 7. \end{cases}$$

$$P(X_2 = j) = \begin{cases} p_{.j}, & j = 1, \dots, 6, \\ p_j, & j = 7. \end{cases}$$

4.2 Model Assumptions

Is it possible to express the joint probabilities for $i = 1, \dots, 6$ and $j = 1, \dots, 6$, in terms of the marginal probabilities, thereby reducing the number of parameters to be estimated to get an estimate of the joint probability distribution of X_1 and X_2? Yes, by making two assumptions. The random variables X_1 and X_2 are obviously dependent. However, in this setting a certain conditional independence assumption seems reasonable. Therefore, our first assumption is, for $i = 1, \dots, 6$ and $j = 1, \dots, 6$, that

$$P(X_1 = i, X_2 = j | X_1 \leq 6,\ X_2 \leq 6)$$
$$= P(X_1 = i | X_1 \leq 6,\ X_2 \leq 6) P(X_2 = j | X_1 \leq 6,\ X_2 \leq 6)$$

or, equivalently,

$$P(X_2 = j | X_2 \leq 6) = P(X_2 = j | X_1 = i,\ X_2 \leq 6).$$

Thus, given that the pigs are not in physical contact with each other the probabilities of the outcomes of the second pig are not affected by the outcome of the first pig. The second assumption we make regards the marginal distributions of X_1 and X_2. We have already concluded that $p_{7.} = p_{.7} = p_7$. We also assume, for $i = j = 1, \dots, 6$, that

$$p_{i.} = p_{.j} = p_i.$$

Thus, the marginal distributions are assumed to be equal. In a practical sense we assume the pigs are identical.

Now, from (1) and the assumptions made on conditional independence and equal marginal distributions, the joint probability distribution of X_1 and X_2 can be written as

$$P(X_1 = i, X_2 = j) = p_{ij} = \begin{cases} p_i p_j / \sum_{k=1}^{6} p_k, & i = 1, \dots, 6, j = 1, \dots, 6, \\ p_i, & i = j = 7. \end{cases} \tag{2}$$

To prove this result, we need to show that $p_{ij} = p_i p_j / \sum_{k=1}^{6} p_k$, for $i = 1, \dots, 6, j = 1, \dots, 6$. We have that

$$\begin{aligned} &P(X_1 = i, X_2 = j) \\ &= P(X_1 \leq 6,\ X_2 \leq 6) P(X_1 = i, X_2 = j | X_1 \leq 6,\ X_2 \leq 6) \\ &= P(X_1 \leq 6,\ X_2 \leq 6) P(X_1 = i | X_1 \leq 6,\ X_2 \leq 6) P(X_2 = j | X_1 \leq 6,\ X_2 \leq 6) \\ &= P(X_1 = i) P(X_2 = j | X_2 \leq 6) \\ &= P(X_1 = i) P(X_2 = j) / P(X_2 \leq 6) \\ &= p_i p_j / \sum_{k=1}^{6} p_k, \end{aligned}$$

where the first step follows from the multiplication law of probability and the fact that $\{X_1 = i, X_2 = j\}$ is a subset of $\{X_1 \leq 6,\ X_2 \leq 6\}$. In the second step we impose the conditional independence assumption, while the third step is valid, once again using the multiplication law of probability, since $\{X_1 = i\}$ is a subset of $\{X_1 \leq 6,\ X_2 \leq 6\}$ and using the fact that $X_1 \leq 6$ with certainty if $X_2 \leq 6$. The fourth step follows from the definition of conditional probability, at the same time noting $\{X_2 = i\}$ is a subset of $\{X_2 \leq 6\}$. Finally, in the fifth step we make use of the assumption of equal marginal distributions.

4.3 A Strategy of Maximizing Expected Score in a Round

As mentioned, there are strategies developed for the non-commercial game analogue Pig, see Neller and Presser (2004), and Shi (2000). In this paper, as in Kern (2006) and Gorman (2012), a strategy analysis inspired by Shi (2000) is conducted and adjusted for the game *Pass the Pigs*®. The strategy is to maximize the expected score in a round. Although this strategy does not fully capture the complexity of the game, the strategy proposed by Neller and Presser (2004) is even better if the objective is to win the game, it is a simple strategy that works pretty well and is easy to implement.

Since the expected score from the next roll decreases with the number of rolls, the strategy of maximizing the expected score in a round is equivalent to the strategy of going on rolling the pigs as long as the expected gain in score from the next roll is positive. Let V be the gain in score from rolling the pigs one more time and let $q_{-(S+T)}, q_{-T}, \dots, q_{60}$ be the probabilities of the possible scores outlined in Table 2. We get the expected gain in score as

$$E(V) = -(S+T)q_{-(S+T)} - Tq_{-T} + q_1 + 5q_5 + 10q_{10} + 15q_{15} + 20q_{20}$$
$$+25q_{25} + 40q_{40} + 60q_{60}.$$

As proposed above we use the strategy of continue rolling the pigs as long as

$E(V) > 0$.

Solving for T yields

$$T < \frac{q_1 + 5q_5 + 10q_{10} + 15q_{15} + 20q_{20} + 25q_{25} + 40q_{40} + 60q_{60} - Sq_{-(S+T)}}{q_{-(S+T)} + q_{-T}}. \quad (3)$$

Thus, in maximizing the expected score in a round keep rolling the pigs as long as the accumulated score in the round is smaller than the right-hand side of (3). For further analysis we would like to express this rule in terms of the elements of $\boldsymbol{p}$. From Table 2 and (2) we can write the elements of $\boldsymbol{q}$ as

$$q_{-(S+T)} = p_7, \quad (4)$$

$$q_{-T} = 2p_1p_2/(1-p_7), \quad (5)$$

$$q_1 = (p_1^2 + p_2^2)/(1-p_7), \quad (6)$$

$$q_5 = (2p_1p_3 + 2p_1p_4 + 2p_2p_3 + 10p_2p_4)/(1-p_7), \quad (7)$$

$$q_{10} = (2p_1p_5 + 2p_2p_5 + 2p_3p_4)/(1-p_7), \quad (8)$$

$$q_{15} = (2p_1p_6 + 2p_2p_6 + 2p_3p_5 + 2p_4p_5)/(1-p_7), \quad (9)$$

$$q_{20} = (p_3^2 + p_4^2 + 2p_3p_6 + 2p_4p_6)/(1-p_7), \quad (10)$$

$$q_{25} = 2p_5p_6/(1-p_7), \quad (11)$$

$$q_{40} = p_5^2/(1-p_7), \quad (12)$$

$$q_{60} = p_6^2/(1-p_7), \quad (13)$$

where we have made use of the constraint $\sum_{i=1}^{7} p_i = 1$ and the addition law of probability. Denote the right-hand side in (3) by θ. Substitution of (4) - (13) yields, after rearranging terms

$$\theta = h(\boldsymbol{p}) = \frac{f(\boldsymbol{p})}{g(\boldsymbol{p})}, \tag{14}$$

where

$$f(\boldsymbol{p}) = p_1^2 + p_2^2 + 20p_3^2 + 20p_4^2 + 40p_5^2 + 60p_6^2 + Sp_7^2 + 10p_1p_3 + 10p_1p_4 + 20p_1p_5$$
$$+30p_1p_6 + 10p_2p_3 + 10p_2p_4 + 20p_2p_5 + 30p_2p_6 + 20p_3p_4 + 30p_3p_5 + 40p_3p_6$$
$$+30p_4p_5 + 40p_4p_6 + 50p_5p_6 - Sp_7$$

and

$$g(\boldsymbol{p}) = p_7 - p_7^2 + 2p_1p_2 - 2p_1p_2p_7$$

The interpretation of (14) is as follows: The integer part of θ is the accumulated score where to stop rolling the pigs in order to maximize the expected score in a round as a function of S and the probabilities of a pig´s landing outcomes, unknown probabilities that have to be estimated in order to estimate θ. However, for simplicity, henceforth the parameter θ is referred to this accumulated score, although the integer part of θ is a more accurate description.

5. Inference for the Model

In this section we start by deriving point estimators for the probabilities of the seven landing outcomes based on the model given in (2). Certain properties of these estimators are derived, properties needed for the Delta method, which is used to get an interval estimator for θ. The Delta method is described, and it is also shown how the Delta method can be applied in our context. An alternative method, a Bootstrap method used for inference for θ is also described. A test for model assumptions is also provided.

5.1 Maximum Likelihood Estimation of the Joint Probability Model

In order to get data to estimate the parameters in the model set out in the previous section, the pair of pigs are rolled n times. To this experiment consisting of n identical and independent trials we can associate the 37 random variables $(N_{11}, N_{12}, \dots, N_{66}, N_{77})$, where N_{ij} takes the value n_{ij} if the number of times a roll results in $\{X_1 = i, X_2 = j\}$ is n_{ij}. Clearly, $\sum_{i=1}^{6}\sum_{j=1}^{6} n_{ij} + n_{77} = n$ and the joint distribution of $(N_{11}, N_{12}, \dots, N_{66}, N_{77})$ is multinomial with parameters n and $(p_{11}, p_{12}, \dots, p_{66}, p_{77})$, where $\sum_{i=1}^{6}\sum_{j=1}^{6} p_{ij} + p_{77} = 1$ and p_{ij} is defined by (2). We can write the likelihood as a function of the parameter vector $\boldsymbol{p} = (p_1, p_2, p_3, p_4, p_5, p_6, p_7)$ as

$$L(\boldsymbol{p}) = \prod_{i=1}^{6}\prod_{j=1}^{6}(p_ip_j/\sum_{k=1}^{6} p_k)^{n_{ij}}\, p_7^{n_{77}}.$$

The log-likelihood function, defined by $l(\boldsymbol{p}) = \ln(L(\boldsymbol{p}))$, is given by

$$l(\boldsymbol{p}) = \textstyle\sum_{i=1}^{6}\sum_{j=1}^{6} n_{ij}[\ln(p_i) + \ln(p_j) - \ln(\sum_{k=1}^{6} p_k)] + n_{77}\ln(p_7).$$

Taking the constraint $\sum_{i=1}^{7} p_i = 1$ into account when maximizing the log-likelihood with respect to $\boldsymbol{p}$ calls for the method of Lagrange multipliers. The Lagrangean function is written as

$$\mathcal{L}(\boldsymbol{p},\lambda) = l(\boldsymbol{p}) + \lambda(1 - \textstyle\sum_{i=1}^{7} p_i).$$

In finding the maximum likelihood estimators for the parameters in the vector $\boldsymbol{p}$ we take the partial derivatives of $\mathcal{L}(\boldsymbol{p},\lambda)$ with respect to $\boldsymbol{p}$ and λ. We obtain

$$\frac{\partial\mathcal{L}(\boldsymbol{p},\lambda)}{\partial p_i} = \frac{\sum_{j=1}^{6}(n_{ij} + n_{ji})}{p_i} - \frac{\sum_{k=1}^{6}\sum_{j=1}^{6} n_{ij}}{\sum_{k=1}^{6} p_k} - \lambda, \qquad i = 1, \dots, 6,$$

$$\frac{\partial\mathcal{L}(\boldsymbol{p},\lambda)}{\partial p_7} = \frac{n_{77}}{p_7} - \lambda$$

and

$$\frac{\partial\mathcal{L}(\boldsymbol{p},\lambda)}{\partial\lambda} = 1 - \sum_{i=1}^{7} p_i$$

Setting the derivatives equal to zero we obtain

$$\frac{\sum_{j=1}^{6}(n_{ij} + n_{ji})}{\hat{p}_i} - \frac{\sum_{k=1}^{6}\sum_{j=1}^{6} n_{ij}}{\sum_{k=1}^{6} \hat{p}_k} = \lambda, \qquad i = 1, \dots, 6, \qquad (15)\text{-}(20)$$

$$\frac{n_{77}}{\hat{p}_7} = \lambda \qquad (21)$$

$$1 - \sum_{i=1}^{7} \hat{p}_i = 0, \qquad (22)$$

where $\hat{p}_i$, $i = 1, \dots, 7$, is the maximum likelihood estimate of p_i. To solve for these estimates we subtract, one at the time, the equations (16) to (20) from (15) and solve for $\hat{p}_i, i = 2, \dots, 6$, in terms of $\hat{p}_1$. This gives

$$\hat{p}_i = \hat{p}_1 \frac{\sum_{j=1}^{6}(n_{ij} + n_{ji})}{\sum_{j=1}^{6}(n_{1j} + n_{j1})}. \qquad i = 2, \dots, 6. \qquad (23)\text{-}(27)$$

Now, subtracting (21) from (15) and substituting $1 - \hat{p}_7$ for $\sum_{k=1}^{6} \hat{p}_k$ we can solve for $\hat{p}_7$ in terms of $\hat{p}_1$. We get

$$\hat{p}_7 = \hat{p}_1 \frac{2n_{77}}{\sum_{j=1}^{6}(n_{1j} + n_{j1})}. \qquad (28)$$

Now, substitute (23) to (28) into (22), i.e. the constraint, to solve for $\hat{p}_1$. This gives, using the constraint $\sum_{i=1}^{6}\sum_{j=1}^{6} n_{ij} + n_{77} = n$,

$$\hat{p}_1 = \frac{\sum_{j=1}^{6}(n_{1j} + n_{j1})}{2n}. \qquad (29)$$

Finally, to solve for $\hat{p}_i, i = 2, \dots, 7$, substitution of (29) into (23) to (28) yields

$$\hat{p}_i = \frac{\sum_{j=1}^{6}(n_{ij} + n_{ji})}{2n}, \qquad i = 1, \dots, 6$$

and

$$\hat{p}_7 = \frac{n_{77}}{n}.$$

Thus, the maximum likelihood estimators $\widehat{\boldsymbol{p}} = (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5, \hat{p}_6, \hat{p}_7)$ of the marginal probabilities are given by

$$\hat{p}_i = \frac{\sum_{j=1}^{6}(N_{ij} + N_{ji})}{2n}, \qquad i = 1, \dots, 6 \tag{30}$$

and

$$\hat{p}_7 = \frac{N_{77}}{n}. \tag{31}$$

The interpretation of the estimators is straightforward and intuitively appealing. To estimate the probability of the outcome that a pig is in physical contact with the other pig we simply calculate the proportion of times this outcome occurs out of the n trials. For a nice interpretation of how to estimate the probabilities of the other six outcomes we rewrite (30) as

$$\hat{p}_i = \frac{2N_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{6}(N_{ij} + N_{ji})}{2n}, \qquad i = 1, \dots, 6, \tag{32}$$

where the last term in the numerator is the number of times, we get outcome i exactly once in the n rolls. Thus, the numerator is the number of times outcome i occurs summed over both pigs, that is, out of $2n$ times. The interpretation of $\hat{p}_i$ as a proportion is now obvious. Yet another interpretation is possible rewriting (30) as

$$\hat{p}_i = \left(\frac{\sum_{j=1}^{6} N_{ij}}{n} + \frac{\sum_{j=1}^{6} N_{ji}}{n}\right)/2, \qquad i = 1, \dots, 6$$

Clearly, to estimate the probability of one of the other six outcomes we calculate the average of the proportion of times this particular outcome occurs for the two pigs.

Due to the invariance property, the maximum likelihood estimators $\widehat{\boldsymbol{p}}' = (\hat{p}_{11}, \hat{p}_{12}, \dots, \hat{p}_{66}, \hat{p}_{77})$ of the joint probabilities $\boldsymbol{p}' = (p_{11}, p_{12}, \dots, p_{66}, p_{77})$ are

$$\hat{p}_{ij} = \begin{cases} \hat{p}_i \hat{p}_j / \sum_{k=1}^{6} \hat{p}_k, & i = 1, \dots, 6, j = 1, \dots, 6, \\ \hat{p}_i, & i = j = 7. \end{cases}$$

These estimators are used in a goodness-of-fit test, presented later.

5.2 Some Properties of $\widehat{\boldsymbol{p}}$

In this section we show that $E(\widehat{\boldsymbol{p}}) = \boldsymbol{p}$ and derive the covariance matrix $\boldsymbol{V}(\widehat{\boldsymbol{p}}_n) = \frac{1}{n}\boldsymbol{V}$, where $\boldsymbol{V}$ is a symmetric 7×7 matrix with elements denoted by σ_{ij}. In doing that we make use of three properties of the multinomial probability distribution. It can be found in many texts, see for example Wackerly et al. (2008), that if the random vector $(Y_1, Y_2, \dots, Y_k)$ follows a multinomial distribution with parameters $p_1, p_2, \dots, p_k$ and n, it holds that

$E(Y_i) = np_i, Var(Y_i) = np_i(1 - p_i)$, and $Cov\big(Y_i, Y_j\big) = -np_ip_j$ if $i \neq j$.

Thus, in our context, we have

$$E\big(N_{ij}\big) = np_{ij}, \tag{33}$$

$$Var\big(N_{ij}\big) = np_{ij}\big(1 - p_{ij}\big) \tag{34}$$

and

$$Cov\big(N_{ij,}\, N_{st}\big) = -np_{ij}p_{st} \;\text{ if } (i, j) \neq (s, t), \tag{35}$$

Starting with the proof of unbiasedness we get from (30), making use of (33), where p_{ij} is given by (2), and well-known rules for handling the expectation operator,

$$E(\hat{p}_i) = E\Big(\frac{\sum_{j=1}^{6}(N_{ij} + N_{ji})}{2n}\Big)$$

$$= \frac{1}{2n} E \sum_{j=1}^{6}(N_{ij} + N_{ji})$$

$$= \frac{1}{2n}\sum_{j=1}^{6}(E(N_{ij}) + E(N_{ji}))$$

$$= \frac{1}{2n}\sum_{j=1}^{6}(np_{ij} + np_{ji})$$

$$= \frac{1}{2}\sum_{j=1}^{6}\Big(\frac{p_ip_j}{\sum_{k=1}^{6} p_k} + \frac{p_jp_i}{\sum_{k=1}^{6} p_k}\Big)$$

$$= \frac{1}{2}\sum_{j=1}^{6}\frac{2p_ip_j}{\sum_{k=1}^{6} p_k}$$

$$= p_i.$$

From (31) we have

$$E(\hat{p}_7) = E\Big(\frac{N_{77}}{n}\Big)$$

$$= \frac{1}{n} E(N_{77})$$

$$= \frac{1}{n} n p_7$$

$$= p_7.$$

Turning to the elements of the covariance matrix, we start with the diagonal elements. In deriving the diagonal elements $\frac{1}{n}\sigma_{ii} = V(\hat{p}_i)$ for $i = 1, \dots, 6$, to simplify the calculations, we consider the way $\hat{p}_i$ is expressed in (32). Define $N_{*i} = \sum_{\substack{j=1 \\ j \neq i}}^{6} (N_{ij} + N_{ji})$, where, as mentioned before, N_{*i} is the number of times we get outcome i exactly once out of the n rolls. From (2) and the properties of the multinomial distribution given previously, i.e. (34) and (35), it follows that

$$V(N_{ii}) = n \frac{p_i^2}{1-p_7} \left(1 - \frac{p_i^2}{1-p_7}\right), \tag{36}$$

$$V(N_{*i}) = n \frac{2p_i(1-p_i-p_7)}{1-p_7} \left(1 - \frac{2p_i(1-p_i-p_7)}{1-p_7}\right), \tag{37}$$

and

$$Cov(N_{ii}, N_{*i}) = -n \frac{p_i^2}{1-p_7} \frac{2p_i(1-p_i-p_7)}{1-p_7}. \tag{38}$$

Now, using the notation for $\sum_{\substack{j=1 \\ j \neq i}}^{6} (N_{ij} + N_{ji})$ introduced in this section as well as well-known rules for the variance operator, we get

$$\begin{aligned} V(\hat{p}_i) &= V\left(\frac{2N_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{6} (N_{ij} + N_{ji})}{2n} \right) \\ &= V\left(\frac{2N_{ii} + N_{*i}}{2n} \right) \\ &= \frac{1}{4n^2} \left(4V(N_{ii}) + V(N_{*i}) + 4Cov(N_{ii}, N_{*i}) \right) \end{aligned} \tag{39}$$

Substitution of (36) to (38) into (39), and simplifying, yields

$$\frac{1}{n}\sigma_{ii} = V(\hat{p}_i) = \frac{1}{n} \frac{p_i(2p_i - 1)p_7 + p_i(1-p_i)}{2(1-p_7)}, \quad i = 1, \dots, 6. \tag{40}$$

Substituting $V(N_{77}) = np_7(1-p_7)$, we get the final diagonal element $\frac{1}{n}\sigma_{77}$ as

$$\begin{aligned} V(\hat{p}_7) &= V\left(\frac{N_{77}}{n} \right) \\ &= \frac{1}{n^2} V(N_{77}) \end{aligned} \tag{41}$$

$$= \frac{p_7(1-p_7)}{n}$$

The off-diagonal elements $\frac{1}{n}\sigma_{ij} = Cov(\hat{p}_i, \hat{p}_j)$ for $i = 1, \ldots, 6$, and $j = 1, \ldots, 6$, where $i \neq j$, is

$$Cov(\hat{p}_i, \hat{p}_j) = Cov\left(\frac{2N_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{6}(N_{ij} + N_{ji})}{2n}, \frac{2N_{jj} + \sum_{\substack{i=1 \\ i\neq j}}^{6}(N_{ji} + N_{ij})}{2n}\right)$$

$$= Cov\left(\frac{2N_{ii} + N_{*i}}{2n}, \frac{2N_{jj} + N_{*j}}{2n}\right) \tag{42}$$

$$= \frac{1}{4n^2}\left(4Cov(N_{ii}, N_{jj}) + 2Cov(N_{ii}, N_{*j}) + 2Cov(N_{jj}, N_{*i}) + 2Cov(N_{*i}, N_{*j})\right)$$

where rules for the covariance operator have been used in the last step. Once again, from the properties of the multinomial distribution we have

$$Cov(N_{ii}, N_{jj}) = -n\frac{p_i^2}{1-p_7}\frac{p_j^2}{1-p_7}, \tag{43}$$

$$Cov(N_{ii}, N_{*j}) = -n\frac{p_i^2}{1-p_7}\frac{2p_j(1-p_j-p_7)}{1-p_7}, \tag{44}$$

and, reversing the subscripts, we get

$$Cov(N_{jj}, N_{*i}) = -n\frac{p_j^2}{1-p_7}\frac{2p_i(1-p_i-p_7)}{1-p_7}. \tag{45}$$

However, these properties of the multinomial distribution cannot directly be used to get an expression for the last covariance term $Cov(N_{*i}, N_{*j})$, since the two outcomes (i, j) and (j, i) are used to calculate N_{*i} as well N_{*j}. Therefore, we define $N_{*ij} = \sum_{\substack{k=1 \\ k\neq i, k\neq j}}^{6} (N_{ik} + N_{jk})$ and $N_{**ij} = N_{ij} + N_{ji}$. We note that $N_{*i} = N_{*ij} + N_{**ij}$ and the interpretation of N_{*ij} and N_{**ij} should be clear. N_{*ij} is the number of times outcome i occurs exactly once without outcome j occurs, while N_{**ij} is the number of times outcomes i and j occur at the same time. Now, using properties of the covariance operator, we get

$$\begin{aligned} Cov(N_{*i}, N_{*j}) &= Cov(N_{*ij} + N_{**ij}, N_{*ji} + N_{**ji}) \\ &= Cov(N_{*ij}, N_{*ji}) + Cov(N_{*ij}, N_{**ji}) + Cov(N_{**ij}, N_{*ji}) + V(N_{**ij}), \end{aligned} \tag{46}$$

where the last term is valid, since the fact that $N_{**ij} = N_{**ji}$ implies $Cov(N_{**ij}, N_{**ji}) = V(N_{**ij})$. By using properties of the multinomial distribution, in a similar way we have made before by using (34) and (35), we have

$$Cov(N_{*ij}, N_{*ji}) = -n\frac{2p_i(1-p_i-p_j-p_7)}{1-p_7}\frac{2p_j(1-p_i-p_j-p_7)}{1-p_7}, \tag{47}$$

$$Cov(N_{*ij}, N_{**ji}) = -n\frac{2p_i(1 - p_i - p_j - p_7)}{1 - p_7}\frac{2p_jp_i}{1 - p_7}, \tag{48}$$

$$Cov(N_{**ij}, N_{*ji}) = -n\frac{2p_ip_j}{1 - p_7}\frac{2p_j(1 - p_i - p_j - p_7)}{1 - p_7}, \tag{49}$$

and

$$V(N_{**ij}) = n\frac{2p_ip_j}{1 - p_7}\left(1 - \frac{2p_ip_j}{1 - p_7}\right). \tag{50}$$

Now, substitute (47) to (50) into (46) to get

$$Cov(N_{*i}, N_{*j}) = \frac{2np_ip_j(1 - 2(1 - p_i - p_j - p_7))}{1 - p_7} - \frac{4np_i^2p_j^2}{(1 - p_7)^2} \tag{51}$$

Finally, substitute (43) to (45) and (51) into (42). After simplification we end up with

$$\frac{1}{n}\sigma_{ij} = Cov(\hat{p}_i, \hat{p}_j) = \frac{1}{n}\frac{p_ip_jp_7 - \frac{1}{2}p_ip_j}{(1 - p_7)}, \tag{52}$$

for $i = 1, \dots, 6$, and $j = 1, \dots, 6$, where $i \neq j$.

We have yet one more case to consider, the off-diagonal elements $\frac{1}{n}\sigma_{i7} = \frac{1}{n}\sigma_{7i} = Cov(\hat{p}_i, \hat{p}_7)$ for $i = 1, \dots, 6$. With a similar motivation as for the previous cases, we get

$$\begin{aligned} Cov(\hat{p}_i, \hat{p}_j) &= Cov\left(\frac{2N_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{6}(N_{ij} + N_{ji})}{2n}, \frac{N_{77}}{n}\right) \\ &= Cov\left(\frac{2N_{ii} + N_{*i}}{2n}, \frac{N_{77}}{n}\right) \\ &= \frac{1}{2n^2}\left(2Cov(N_{ii}, N_{77}) + Cov(N_{*i}, N_{77})\right). \end{aligned} \tag{53}$$

It is easily shown that

$$Cov(N_{ii}, N_{77}) = -n\frac{p_i^2}{1 - p_7}p_7 \tag{54}$$

and

$$Cov(N_{*i}, N_{77}) = -n\frac{2p_i(1 - p_i - p_7)}{1 - p_7}p_7. \tag{55}$$

Substituting (54) and (55) into (53), after simplification, we get

$$\frac{1}{n}\sigma_{i7} = \frac{1}{n}\sigma_{7i} = Cov(\hat{p}_i, \hat{p}_7) = -\frac{p_i p_7}{n}. \tag{56}$$

5.3 Inference for the Strategy of Maximizing Expected Score

Recall equation (14) showing θ, where the integer part of θ is the number of points to obtain in a round before stop rolling the pigs, as a function of $\boldsymbol{p}$ and S. Now that we have derived maximum likelihood estimators for $\boldsymbol{p}$, we have, again using the invariance principle, a maximum likelihood estimator for θ as well, simply by replacing $\boldsymbol{p}$ by $\widehat{\boldsymbol{p}}$ given in (30) and (31) to obtain the point estimator

$$\hat{\theta} = h(\widehat{\boldsymbol{p}}) = \frac{f(\widehat{\boldsymbol{p}})}{g(\widehat{\boldsymbol{p}})}.$$

To find interval estimators we make use of the Multivariate Delta Method as well as Bootstrap methods. Starting with the Delta Method consider the following theorem (Casella and Berger, 2002):

Theorem (Multivariate Delta Method)

Let $\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_n$ be a random sample with $E(Y_{ik}) = \mu_i$ and $Cov\left(Y_{ik}, Y_{jk}\right) = \sigma_{ij}$. For a given function h with continuous first partial derivatives and a specific value of $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$ for which $\tau^2 = \sum_{i=1}^{r}\sum_{j=1}^{r}\sigma_{ij}\frac{\partial h(\boldsymbol{\mu})}{\partial \mu_i}\frac{\partial h(\boldsymbol{\mu})}{\partial \mu_j} > 0$,

$$\sqrt{n}[h(\bar{Y}_1, \dots, \bar{Y}_r) - h(\mu_1, \dots, \mu_r)] \longrightarrow N(0, \tau^2)$$

in distribution, where $\bar{Y}_i = \frac{1}{n}\sum_{k=1}^{n} Y_{ik}$.

To apply this theorem on our problem of interest we define for the kth roll, where $k = 1, \dots, n$,

$$Y_{ik} = \frac{2I_{iik} + \sum_{\substack{j=1 \\ j\neq i}}^{6}(I_{ijk} + I_{jik})}{2}, \qquad i = 1, \dots, 6,$$

and

$$Y_{ik} = I_{iik}, \qquad i = 7,$$

where I_{ijk} is an indicator variable taking the value 1 if, for the kth roll, the outcome of the first pig is i and the outcome of the second pig is j. Now, $E(Y_{ik}) = p_i$ and $Cov\left(Y_{ik}, Y_{jk}\right) = \sigma_{ij}$, where σ_{ij} is given from (40), (41), (52) and (56) in the previous section. It is also clear that $\hat{p}_i$ can be expressed as $\frac{1}{n}\sum_{k=1}^{n} Y_{ik}$. Thus, the above theorem can be used. For a large sample size n, we have

$$\hat{\theta} \overset{appr}{\sim} N\left(\theta, \frac{1}{n}\sum_{i=1}^{r}\sum_{j=1}^{r}\sigma_{ij}\frac{\partial\theta}{\partial p_i}\frac{\partial\theta}{\partial p_j}\right),$$

where the partial derivatives are given below.

For $(i,j) = (1,2),(2,1)$,

$$\frac{\partial\theta}{\partial p_i} = \frac{(2p_i + 10p_3 + 10p_4 + 20p_5 + 30p_6)g(\boldsymbol{p}) - (2p_j - 2p_j p_7)f(\boldsymbol{p})}{(g(\boldsymbol{p}))^2}.$$

For $(i,j) = (3,4),(4,3)$,

$$\frac{\partial\theta}{\partial p_i} = \frac{40p_i + 10p_1 + 10p_2 + 20p_j + 30p_5 + 40p_6}{g(\boldsymbol{p})},$$

while for $i = 5,6,7$,

$$\frac{\partial\theta}{\partial p_5} = \frac{80p_5 + 20p_1 + 20p_2 + 30p_3 + 30p_4 + 50p_6}{g(\boldsymbol{p})},$$

$$\frac{\partial\theta}{\partial p_6} = \frac{120p_6 + 30p_1 + 30p_2 + 40p_3 + 40p_4 + 50p_5}{g(\boldsymbol{p})},$$

$$\frac{\partial\theta}{\partial p_7} = \frac{S(2p_7 - 1)g(\boldsymbol{p}) - (2p_7 p_1 p_2 - 1)f(\boldsymbol{p})}{(g(\boldsymbol{p}))^2}.$$

An approximate $100(1-\alpha)$ % confidence interval estimator for θ is now given by

$$(\hat{\theta} - z_{\alpha/2}\frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{\sigma}_{ij}\widehat{\frac{\partial\theta}{\partial p_i}}\widehat{\frac{\partial\theta}{\partial p_j}}}, \hat{\theta} + z_{\alpha/2}\frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{\sigma}_{ij}\widehat{\frac{\partial\theta}{\partial p_i}}\widehat{\frac{\partial\theta}{\partial p_j}}}),$$

where $z_{\alpha/2}$ is the $100(1-\alpha/2)$th percentile for the standard normal distribution, while $\hat{\sigma}_{ij}$ and $\widehat{\frac{\partial\theta}{\partial p_i}}$ are estimators for σ_{ij} and $\frac{\partial\theta}{\partial p_i}$, respectively. To get these estimators, $\boldsymbol{p}$ is simply replaced by $\hat{\boldsymbol{p}}$ in the expressions for σ_{ij} and $\frac{\partial\theta}{\partial p_i}$.

Bootstrap methods are alternatives to the Delta method, see for example the pioneering work by Efron and Tibshirani (1994) for a nice treatment of bootstrap methods. Here we adopt a parametric bootstrap method to obtain an interval estimator for θ. Based on the maximum likelihood estimates of p_i 1000 resamples, each with a size of 5000, are generated using the model described in (7). For each resample an estimate of θ is calculated, this value is denoted by $\hat{\theta}_i^*$ for the ith resample. A bootstrap estimator for the variance of $\hat{\theta}$ is

$$Var^*(\hat{\theta}) = \frac{1}{1000-1}\sum_{i=1}^{1000}(\hat{\theta}_i^* - \overline{\hat{\theta}_i^*})^2,$$

where $\overline{\hat{\theta}_i^*} = \frac{1}{1000}\sum_{i=1}^{1000}\hat{\theta}_i^*$, the mean of the resampled values. An approximate $100(1-\alpha)$ % confidence interval estimator for θ based on bootstrap is now given by

$$(\overline{\hat{\theta}_i^*} - z_{\alpha/2}\sqrt{Var^*(\hat{\theta})}, \overline{\hat{\theta}_i^*} + z_{\alpha/2}\sqrt{Var^*(\hat{\theta})}).$$

As an alternative, an interval based on percentiles of the distribution of the resampled estimates of θ can be utilized, where the $100\alpha/2$th percentile is the lower limit and the $100(1-\alpha/2)$th percentile is the upper limit.

5.4 A Test of Model Assumptions

In this section a goodness-of-fit test, see Wackerly et al. (2008), is proposed for simultaneously testing the assumptions of conditional independence and equal marginal distributions. Note that the two model assumptions imply the joint probability distribution given by (2). Therefore, to test the null hypothesis

H_0: The two model assumptions are fulfilled

against the alternative

H_1: At least one of the two model assumptions is not fulfilled

a goodness-of-fit test can be performed of the null hypothesis

H_0: The joint distribution of X_1 and X_2 is given by (2)

against the alternative

H_1: The joint distribution of X_1 and X_2 is not given by (2).

Recall that under the null hypothesis the joint distribution of $(N_{11}, N_{12}, \dots, N_{66}, N_{77})$ is multinomial with parameters n and $(p_{11}, p_{12}, \dots, p_{66}, p_{77})$ given in (2). Thus, the marginal distribution of N_{ij} are binomial with parameters n and p_{ij}, and $E(N_{ij}) = np_{ij}$. Since p_{ij} is not known we replace the unknown parameter by its maximum likelihood estimator $\hat{p}_{ij}$ to obtain $\widehat{E(N_{ij})} = n\hat{p}_{ij}$, where $\hat{p}_{ij} = \hat{p}_i\hat{p}_j/\sum_{k=1}^{6}\hat{p}_k$, for $i = 1, \dots, 6, j = 1, \dots, 6$, and $\hat{p}_{77} = \hat{p}_7$ under H_0. The test statistic given by

$$\mathrm{X}^2 = \sum_{i=1}^{6}\sum_{j=1}^{6}\frac{(N_{ij} - \widehat{E(N_{ij})})^2}{\widehat{E(N_{ij})}} + \frac{(N_{77} - \widehat{E(N_{77})})^2}{\widehat{E(N_{77})}},$$

has approximately a χ^2 distribution with $(37 - (7-1) - 1) = 30$ degrees of freedom. The significance level to be used is 5 %.

To motivate the number of degrees of freedom associated with this χ^2 statistic recall that this number is equal to the number of cell probabilities minus 1 degree of freedom for each independent restriction put on the cell probabilities. In total we have 37 cell probabilities. We lose 1 degree of freedom because of the constraint $\sum_{i=1}^{6}\sum_{j=1}^{6} p_{ij} + p_{77} = 1$. Moreover, we use the cell frequencies to estimate 7 parameters, $p_i, i = 1, \dots, 7$. However, the seventh parameter is determined once we have estimates for the first six parameters. Thus, we lose yet another $(7-1) = 6$ degrees of freedom for estimation of the parameters.

The sharp-eyed reader might already have noticed that the test statistic given above can be simplified to

$$\sum_{i=1}^{6}\sum_{j=1}^{6}\frac{(N_{ij}-\widehat{E(N_{ij})})^2}{\widehat{E(N_{ij})}},$$

since $\frac{(N_{77}-\widehat{E(N_{77})})^2}{\widehat{E(N_{77})}}$ equals zero under the null hypothesis. The reasoning behind the degrees of freedom being 30 now goes like this: There are 36 cell probabilities and now we estimate 6 parameters, $p_i, i = 1, \ldots, 6$, where no parameter is determined once the others have been estimated. furthermore, no other constraint is placed on the cell probabilities. Thus, we get the number of degrees of freedom as $(36 - 6) = 30$.

6. Results from Estimation and Testing

In Table 4 the maximum likelihood estimates of the marginal distribution of X is presented.

Table 4. Maximum likelihood estimates of the marginal distribution of X.

$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$	$\hat{p}_4$	$\hat{p}_5$	$\hat{p}_6$	$\hat{p}_7$
0.2971	0.3435	0.0956	0.2271	0.0284	0.0045	0.0038

Using the above maximum likelihood estimates for the goodness of fit test set out earlier yields an observed value of the test statistic equal to 34.09 with a p-value of 0.28. Thus, our model is not rejected at the prescribed significance level of 5 %.

For $S = 0$ we get the point estimate $\hat{\theta} = 22.91$, suggesting rolling again if $T \leq 22$. The estimated variance of $\hat{\theta}$ using the Delta method is 0.4451 and 0.4300 using the Bootstrap method. Table 5 below shows 95 percent confidence interval for θ for the three different interval methods used.

Table 5. 95 % confidence intervals for θ setting $S = 0$.

Method	**Interval estimate**
Delta	$(21.60, 24.22)$
Bootstrap using normal approximation	$(21.62, 24.26)$
Bootstrap using percentiles	$(21.61, 24.31)$

The intervals are quite similar, suggesting the normal approximation is a reasonable assumption.

For $S = 60$ the point estimate of θ is only slightly smaller than for $S = 0$. We get $\hat{\theta} = 21.84$ and a 95 % confidence interval $(20.75, 24.00)$. The reason for the small difference in point estimates is of course the small probability of the outcome that the pigs touch each other.

7. Discussion

Apparently, the game *Pass the Pigs®* offers students in statistics a variety of possibilities in training their skills on different concepts and methods usually contained in a typical advanced undergraduate or first year graduate course in Statistical theory. In solving the problem of estimating a confidence interval for the accumulated score where you are supposed to stop rolling the pigs in order to maximize the expected score in a given round, initially students need to think of concepts such as random experiments, outcomes and sample spaces. Furthermore, in formulating a model to be estimated and tested they practice, among other things, the concepts of joint probability distributions, statistical independence and conditional probability as well as the multiplication law of probability. Moreover, in developing a game's strategy they encounter the mathematical expectation as a tool for decision making. In the latter part of the problem the students meet the main inferential methods such as point and interval estimation as well as hypothesis testing. The method of maximum likelihood is used for point estimation while two methods are proposed for interval estimation, the Delta method and the Bootstrap method. Hypothesis testing is used to check for model adequacy in terms of a goodness-of-fit test. Thus, a lot of probabilistic and statistical stuff are used to solve the problem. Therefore, the problem is suitable to serve as a comprehensive home assignment at the end of the course to tie together all the topics covered. Another approach is to split up the problem into several parts, given as smaller home assignments as concepts and methods are introduced during the course.

We are certainly not limited to the approach outlined in this paper to solve the problem. From the results section it is suggested that the unlikely outcome where the pigs touch each other has a small effect on the strategy. Therefore, one possibility is to skip that outcome. Things would simplify a lot, especially those parts contained in the sections of model assumptions and maximum likelihood estimation.

The assumptions of conditional independence and identical marginal distributions outlined earlier are of course not crucial for the construction of a confidence interval. Therefore, another solution of the problem might be obtained if the two assumptions previously mentioned are skipped, either because of rejection of the null hypothesis in section 5.4 or because a simpler model is preferred. In this case the ten parameters $q_{-(S+T)}, q_{-T}, \dots, q_{60}$ are simply estimated by the maximum likelihood estimates m_k/n, where m_k is the number of times the n rolls have resulted in k points, $k = -(S+T), -T, \dots, 60$. Doing the necessary modifications of the formulas yields the point estimate 22.37 for θ and a corresponding interval estimate $(20.75, 24.00)$ with a $95\ \%$ confidence level using the Delta method. This interval is wider than the intervals calculated based on our assumptions of conditional independence and identical marginal distributions, suggesting that provided these assumptions are fulfilled the estimator for θ based on these assumptions is more efficient.

Our point estimates of θ, suggesting rolling until $T \leq 22$, are in line with the result obtained by Kern (2006). However, this similarity in estimation results should be interpreted with care, since variation in rolling techniques and a possible variation in the pigs' shape might influence the true parameter θ, see Kern (2006) for a discussion of various sources of variation. Therefore, a good idea might be to let the groups of students generate their own data to provide the opportunity of testing for these possible sources of variation using homogeneity tests, i.e. testing for equality of distributions among different groups of students controlling for the same pair of pigs as well as testing for equality of the pigs' shape, where rolling techniques are controlled for.

The approach presented in this paper is purely classical. A suggestion for further analysis is to go on with the Bayesian approach suggested by Kern (2006) to get a credible interval for θ. Such an approach would allow for an appealing interpretation like statements of the type: “The probability is 0.67 that you should go on rolling as long as the accumulated score is 22, while the probability is 0.13 you should go on as long as the score is 23”. Such statements are of course not valid using the classical approach.

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