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Matrix Gamma Distributions and Related Stochastic Processes

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MATRIX GAMMA DISTRIBUTIONS AND RELATED STOCHASTIC PROCESSES

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Abstract. There is considerable literature on matrix-variate gamma distributions, also known as Wishart distributions, which are driven by a shape parameter with values in the (Gindikin) set $\{i/2, i = 1, \dots, k-1\} \cup ((k-1)/2, \infty)$. We provide an extension of this class to the case where the shape parameter may actually take on any positive value. In addition to the well-known singular Wishart as well as non-singular matrix-variate gamma distributions, the proposed class includes new singular matrix-variate distributions, with the shape parameter outside of the Gindikin set. This singular, non-Wishart case is no longer permutation invariant and derivation of its scaling properties requires special care. Among numerous newly established properties of the extended class are group-like relations with respect to the positive shape parameter. The latter provide a natural substitute for the classical convolution properties that are crucial in the study of infinite divisibility. Our results provide further clarification regarding the lack of infinite divisibility of Wishart distributions, a classical observation of Paul Lévy. In particular, we clarify why the row/column vectors in the off-diagonal blocks are infinitely divisible. A class of matrix-variate Laplace distributions arises naturally in this set-up as the distributions of the off-diagonal blocks of random gamma matrices. For the class of Laplace rectangular matrices, we obtain distributional identities that follow from the role they play in the structure of the matrix gamma distributions. We present several elegant and convenient stochastic representations of the discussed classes of matrix-valued distributions. In particular, we show that the matrix-variate gamma distribution is a symmetrization of the triangular Rayleigh distributed matrix – a new class of the matrix variables that naturally extend the classical univariate Rayleigh variables. Finally, a connection of the matrix-variate gamma distributions to matrix-valued Lévy processes of a vector argument is made. Namely, a Lévy process, termed a matrix gamma-Laplace motion, is obtained by the subordination of the triangular Brownian motion of a vector argument to a vector-valued gamma motion of a vector argument. In this context, we introduce a triangular matrix-valued Rayleigh process, which, through symmetrization, leads to a new matrix-variate gamma process. This process when taken at a properly defined one-dimensional argument has the matrix gamma marginal distribution with the shape parameter equal to its argument.

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1. Introduction

The classical $k \times k$ matrix-variate gamma (MG) distributions (see, e.g, [15]) involve a matrix scale parameter and a shape parameter, which is restricted to the values above $(k-1)/2$. This class is closed with respect to the convolution, as the sum of two independent MG random matrices is again MG, with the shape parameter being the sum of the shapes of the summands (provided that the scaling matrices of the summands are the same). It is well known that Wishart distributions are a subclass of matrix gamma distributions in an exact analogy with the one-dimensional case, where chi-square distribution arises as a special case of the gamma distribution. The Wishart distributions can be naturally defined also in the singular case (see, e.g., [41]) and the convolution property is also valid for the non-singular matrix-variate gamma distributions accompanied by the singular Wishart distributions. This is a nontrivial result, due to [13], that the convolution property holds over the so-called Gindikin semigroup $\{i/2, i = 1, \dots, k-1\} \cup ((k-1)/2, \infty)$, see also [23]. However, this is as far as one can go regarding the closure with respect to the convolution, which is due to the lack of infinite divisibility for the entire class of MG distributions. Indeed, MG distribution is not infinitely divisible in the usual sense, as remarked by [4]: *"It is relevant to mention that the most common examples in statistics of laws of positive definite random matrices, such as the Wishart and gamma matrix distributions, are not infinitely divisible"*. This was perhaps first noted for the two-dimensional Wishart case in [27]. In [1], it was shown that no Wishart distribution with one degree of freedom is a convolution of two nontrivial distributions. In a far more complete fashion, the issue of infinite divisibility was treated in [38] and [39] for the Wishart random matrix as well as its sub-matrices, where the question of infinite divisibility of the blocks of Wishart random matrices was posed. Further studies of Wishart characteristics functions in [11] and [33] shed more light on the reasons for the lack of infinite divisibility.

In this work, we discuss a natural extension of the classical MG distributions, where the shape parameter is no longer restricted to the Gindikin set and can take on any positive value. The extension includes both the non-singular matrix-variate gamma and singular Wishart distributions, as special cases. The original MG distributions are embedded here without any change (with the shape parameter above $(k-1)/2$), while the singular Wishart family is driven by the shape parameter values in the set $\{1/2, \dots, (k-1)/2\}$. The extension allows for the shape parameter to be anywhere in the interval $(i/2, (i+1)/2]$ as well where $i = 0, \dots, k-2$, with the corresponding rank of the random matrix taking on the values of $i+1$. To the best of our knowledge, the presented extension of matrix-variate gamma distributions to the case

$$\alpha \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \cup \dots \cup \left(\frac{k-2}{2}, \frac{k-1}{2}\right),$$

is the first of its kind. This extension and its properties is among the main contributions of this work. Let us note that while the term "singular gamma" matrix-variate distribution has recently appeared in [31], the singular matrix-variate gamma distributions studied there were actually the familiar singular Wishart distributions, with the shape parameter $\alpha \in \{i/2, i = 1, \dots, k-1\}$, which is the "discrete" part of the Gindikin set.

Even though the classical infinite divisibility group property does not hold for this extended class of distributions, we provide a certain alternative to the standard convolution group property, leading to a closely related divisibility property. We also present a complete characterization of the blocks of random Wishart and gamma matrices in terms of their infinite divisibility. Through this, we argue that the lack of infinite divisibility, which has been pointed out as a problem when building stochastic models based on these distributions, is not necessarily as prohibitive as originally believed. The novel form of divisibility introduced in this paper, while not completely yielding infinite divisibility of the entire matrix, can still be quite effective in the construction of matrix-variate stochastic measures and stochastic integration, which are crucial for time or/and space-dependent models, although this topic is left for future studies. Our results indicate that, instead of reaching into alternative versions of MG infinitely divisible distributions, as proposed in [35] and [4], one can still work with the classical ones and exploit their explicit, analytically tractable properties. Additionally, through the derived stochastic representations of the MG distribution, we address the interpretability of the corresponding random matrices where the shape parameter is not a half-integer, the lack of which was sometimes pointed out in the literature, see [23].

We extend the fundamental analytical properties of the non-singular MG distributions to the singular case, discussed in detail in this paper. In particular, we derive several interesting stochastic representations of random gamma matrices and utilize MG distributions supported on non-negative definite matrices to construct covariance-mean mixtures of matrix-variate normal distributions. This leads to the new class of matrix-variate asymmetric distributions that are termed *matrix-variate generalized asymmetric Laplace* distributions or, in short, matrix asymmetric Laplace (MAL) distributions.

We note that the theory of matrix-variate distributions obtained by mixing a matrix-variate Gaussian distribution with a randomly distributed covariance matrix is well developed. Indeed, [5] considered a new class of infinitely divisible covariance mixtures of Gaussian random matrices, while [35] introduced a class of infinitely divisible positive-definite gamma random matrices and used them to define a matrix gamma-normal distribution, which is also infinitely divisible. In turn, [42] introduced a matrix-variate generalized hyperbolic distribution by compounding matrix-variate normal distribution with generalized inverse Gaussian distributed scale matrix. As a special case of these, the MAL distributions introduced in this paper are fundamentally related to MG distributions, including their representation as the difference of two independent gamma distributed matrices. The close relation between the MAL and MG distributions goes in both directions, as the MAL distribution naturally appears as the distribution of the off-diagonal blocks of gamma distributed matrices (regardless of whether the latter are singular or not).

Our paper is structured as follows. In Section 2, we review basic properties of the classical MG distributions, including the singular case of Wishart distributions. Here, we also discuss the class of MAL distributions as the distributions of the off-diagonal blocks of random gamma matrices and establish some of their fundamental properties that follow from the relation between these two classes. The singular MG distributions are introduced in Section 3, where we discuss their structure and some analytical and distributional results. This is followed by Section 4, where we introduce an exchangeable version of MG distribution with the shape parameter falling below $(k-1)/2$, where k is the matrix dimension. Some basic properties and the relation to the singular case are provided in this section as well. The problem of infinite divisibility of the MG distributions is discussed in Section 5. Here, we discuss the convolution properties of these distributions and address the question posed by [39] concerning the infinite divisibility of the blocks of Wishart and MG distributions. We provide a complete answer to the question through a stochastic process representation of MG distributions. All proofs and technical results are collected in the Appendix.

2. Matrix-variate gamma, Laplace, and singular Wishart distributions

In this section, we collect some known but relevant properties of the non-singular MG and singular Wishart distributions. For this, we need some notation with which we start. We let $\mathbf{I}_k$ denote a $k \times k$ dentity matrix, while $\mathbf{0}$ stands for the suitable matrix of zeros. We use $\mathrm{etr}\{\mathbf{A}\}$ for the exponent of the trace of the matrix $\mathbf{A}$. The symbol $\stackrel{d}{=}$ stands for the equality in distribution and $\otimes$ denotes the Kronecker product. The notation $\mathbf{C} \geq 0$ means that the square matrix $\mathbf{C}$ is non-negative definite and it is positive definite when $\mathbf{C} \geq 0$ and $|\mathbf{C}| > 0$, which will be written as $\mathbf{C} > 0$. The set of all positive definite (symmetric) $k \times k$ matrices is denoted by $\mathbb{S}_k^+$. This set constitutes a cone, that is, it is closed under the addition and multiplication by a positive scalar. The non-negative definite matrices form a cone as well, which is the closure $\overline{\mathbb{S}_k^+}$.

2.1. Non-singular matrix-variate gamma distribution.

Recall that a $k \times k$ positive definite random matrix $\mathbf{X}$ is said to follow a MG distribution, which is denoted by $\mathbf{X} \sim \mathcal{MG}_k(\alpha, \mathbf{A})$, if its probability density function (PDF) is given by

$$f(\mathbf{X}) = \frac{1}{\Gamma_k(\alpha)|\mathbf{A}|^{\alpha}} |\mathbf{X}|^{\alpha-(k+1)/2} \mathrm{etr}\{-\mathbf{A}^{-1}\mathbf{X}\}, \quad \mathbf{X} > 0, \tag{2.1}$$

where $\mathbf{A} \in \mathbb{S}_k^+$ is a scale parameter matrix (also known as a dispersion matrix), α is a shape parameter such that $\alpha > (k-1)/2$, and $\Gamma_k(\alpha)$ is the generalized gamma function, defined by

$$\Gamma_k(\alpha) = \pi^{k(k-1)/4} \prod_{i=1}^{k} \Gamma\left(\alpha - \frac{i-1}{2}\right),$$

see [15, Chapter 3.6]. The crucial condition $\alpha > (k-1)/2$ ensures that the PDF is properly defined and integrates to one over the cone $\mathbb{S}_k^+$. The case with $\mathbf{A} = \mathbf{I}_k$ is referred to as the *standard matrix-variate gamma* distribution of dimension k with shape parameter α, denoted by $\mathcal{MG}_k(\alpha)$.

The LT of $\mathcal{MG}_k(\alpha, \mathbf{A})$ distribution is given by

$$\begin{aligned}\psi_{\mathbf{X}}(\mathbf{T}) &= \frac{1}{\Gamma_k(\alpha)|\mathbf{A}|^{\alpha}} \int_{\mathbf{X}>0} |\mathbf{X}|^{\alpha-(k+1)/2} \mathrm{etr}\{-(\mathbf{A}^{-1}+\mathbf{T})\mathbf{X}\} \mathbf{dX} \\ &= |\mathbf{I}_k + \mathbf{TA}|^{-\alpha},\end{aligned} \tag{2.2}$$

where $\mathbf{T}$ is a $k \times k$ symmetric matrix such that $\mathbf{A}^{-1} + \mathbf{T}$ (or, equivalently, $\mathbf{A}(\mathbf{I}_k + \mathbf{TA})$ or $\mathbf{I}_k + \mathbf{A}^{\frac{1}{2}}\mathbf{TA}^{\frac{1}{2}}$) is positive definite. In particular, the Laplace transform is well defined for all positive definite $\mathbf{T}$, which is true for any distribution on the cone $\mathbb{S}_k^+$.

This result about the domain of the LT is easily obtained by scaling the density of a MG distribution and noticing that

$$|\mathbf{A}^{-1} + \mathbf{T}| = |\mathbf{A}^{-1}||\mathbf{I}_k + \mathbf{A}\mathbf{T}| = |\mathbf{A}^{-1}||\mathbf{I}_k + \mathbf{T}\mathbf{A}|.$$

Similarily, the characteristic function (ChF) of (2.1) is given by

$$\varphi_{\mathbf{X}}(\mathbf{T}) = |\mathbf{I}_k - \iota\mathbf{T}\mathbf{A}|^{-\alpha}, \tag{2.3}$$

where $\iota^2 = -1$.

Remark 1. *In general, the LT and the ChF of a random* $k \times n$ *matrix* $\mathbf{X}$ *are given by* $\psi_{\mathbf{X}}(\mathbf{T}) = \mathbb{E}\left[\operatorname{etr}\{-\mathbf{T}^\top\mathbf{X}\}\right]$ *and* $\varphi_{\mathbf{X}}(\mathbf{T}) = \mathbb{E}\left[\operatorname{etr}\{\iota\mathbf{T}^\top\mathbf{X}\}\right]$*, respectively, where* $\mathbf{T}$ *is a* $k \times n$ *real matrix (in the case of the Laplace transform,* $\mathbf{T}$ *must be also such that the expectation is well-defined). However, here we can assume that the argument* $\mathbf{T} = (t_{ij})$ *is a symmetric matrix. Indeed, since* $\mathbf{X}$ *is a symmetric matrix, the trace of* $\mathbf{T}^\top\mathbf{X}$ *depends on* t_{ij} *and* t_{ji} *only through* $t_{ij} + t_{ji}$*, i.e.*

$$\operatorname{tr}(\mathbf{T}^\top\mathbf{X}) = \operatorname{tr}\left(\frac{\mathbf{T} + \mathbf{T}^\top}{2}\mathbf{X}\right). \tag{2.4}$$

Thus, the assumption of the symmetry of $\mathbf{T}$ *is not restrictive. However, it is important to note that the final expressions in (2.2) and (2.3) are valid only for symmetric* $\mathbf{T}$.

Remark 2. *It is customary to use the Laplace transform for non-negative variables. This tradition has been extended for distributions on cones, such as* $\mathbb{S}_k^+$ *(see* [34]*). For a given distribution, the domain of the LT is the distribution specific even in the one dimensional case, although it always contains* $[0, \infty)$*. Similarly, for the distribution on the cone* $\mathbb{S}_k^+$*, it always contains the cone itself. The question of whether the LT restricted to the values in* $\mathbb{S}_k^+$ *determines the distribution on that cone does not appear to be fully resolved, as we could find no results equivalent to Lerch's theorem in one dimension,* [22]*. Even in the case of infinitely divisible distributions the typical argument starts with Fourier based Kchintchine-Lévy representation, see, for example,* [34]*. However, the sufficiency of the LT restricted to the cone for a unique definition of the Fourier transform, and thus of the infinitely divisible distribution, does not seem to have been addressed so far. In the literature, see for example* [26] *and* [32]*, the case of the non-singular gamma distribution expanded by the singular Wishart distributions is defined through the form of the LT on the cone, even though the considered distributions are not infinitely divisible. Thus correctness of such a definition is not obvious. In general, the fact that the LT with the argument restricted to* $\mathbb{S}_k^+$ *defines the distribution of a positive-definite random matrix should be probably argued through the analytic extension of such a function, see* [18] *for a discussion of the LT and its inverse with the argument being a matrix with complex entries. Nevertheless, we could not find any explicit references regarding this fact and thus we prefer to utilize the properties of the Fourier transform (characteristic function), which avoids this issue. We note that* [40] *is sometimes quoted as a source of the theory on the Laplace transform on cones, see, for example,* [35]*. However, the book actually uses the ChFs to deliver the result on the canonical Lévy-Kchintchine representation of the infinitely divisible distribution on a cone in a Euclidian space. Consequently, it is not easy, at least for us, to find proper reference to the mathematical theory of the Laplace transform for distributions on a cone. Due to all this, we share the sentiment of William Feller who* *said 'Within probability theory the use of multidimensional (Laplace) transforms is comparatively restricted', see XIII.8* [12].

There are other aspects that make the ChF of the matrix valued argument a more convenient tool than the LT restricted to $\mathbb{S}_k^+$. *While for non-negative random vectors on* $\mathbb{R}^k$ *the LT restricted to the cone of vectors with non-negative entries does define the distribution uniquely (see, e.g.,* [28]*), this may not be so for arbitrary distributions on* $\mathbb{R}^k$. *Additionally, when we move to the cone of* $k \times k$ *positive definite matrices, there is a difficulty with obtaining the distributions of sub-blocks when working with the LT limited in the domain to the cone of positive definite matrices. For example, consider the following general structure of a positive definite random matrix with independent variables* Γ, *standard normal* Z, *and* $\tilde{\Gamma}$ *that is also considered in this paper,*

$$\mathbf{X} = \begin{pmatrix} \Gamma & \sqrt{\Gamma} Z \\ \sqrt{\Gamma} Z & Z^2 + \tilde{\Gamma} \end{pmatrix},$$

with the two cases of Γ*: (1) standard exponential and (2) the reciprocal of squared standard normal. Then, the Laplace transform* $\Psi_{\mathbf{X}}(\mathbf{T})$ *is well defined for* $\mathbf{T} \in \mathbb{S}_2^+$. *The off-diagonal term* $Y = \sqrt{\Gamma} Z$ *has Cauchy distribution in case (2) and the standard Laplace distribution in case (1). However, in the Cauchy case, this term does not have a well defined LT while in the Laplace case, the LT is not properly defined when the argument is outside of* $(-1, 1)$. *Obtaining the LT of these off-diagonal distributions directly from* $\Psi_{\mathbf{X}}(\mathbf{T})$, $\mathbf{T} = (t_{ij}) \in \mathbb{S}_2^+$, *is not possible. Indeed, one cannot simply take* $t_{11} = t_{22} = 0$ *and then check the resulting LT, because such* $\mathbf{T}$ *is not positive definite. Thus, the relation between the LT of the full matrix and its blocks becomes complicated. This highlights technical difficulties of using the LT on the cone. On the other hand, using it beyond the cone is problematic as well, as it may be difficult to accurately define the domain. All this make the ChF on the domain of all symmetric matrices a more 'worry-free' technical tool, which we shall utilize in this paper. However, to keep up with the tradition, we also formulate some of the results using the LT, in which case we also provide the regions of symmetric matrix argument where the formulas hold, going beyond the cone, in a similar way as it has been presented in the text following* (2.2).

Remark 3. *Let us note that if* n *is a positive integer and* $n \geq k$, *then the* $\mathcal{MG}_k(\alpha, \mathbf{A})$ *distribution with* $\alpha = n/2$ *and* $\mathbf{A} = 2\mathbf{\Sigma}$ *coincides with the Wishart distribution with* n *degrees of freedom, see, e.g.,* [15, Chapter 3]. *This is the distribution of* $\mathbf{X}\mathbf{X}^\top$, *where* $\mathbf{X}$ *is a* $k \times n$ *random matrix whose columns are independent and identically distributed (IID) mean-zero multivariate normal vectors with covariance matrix* $\mathbf{\Sigma}$.

Remark 4. *When* $k = 1$, *where we have* $\alpha > (k-1)/2 = 0$ *and* $\mathbf{A} = a > 0$, *the MG* $\mathcal{MG}_k(\alpha, \mathbf{A})$ *distribution reduces to a familiar univariate gamma distribution with shape parameter* α *and scale parameter* $\beta = 1/a$, *denoted by* $\mathcal{G}(\alpha, \beta)$ *and given by the PDF*

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \;\; x \in \mathbb{R}_+. \tag{2.5}$$

In the standard MG case with $k = 1$, *we get what we call standard gamma distribution with shape* α *(and scale 1).*

Remark 5. *It should be noted that if the argument $\mathbf{T}$ of the LT in (2.2) is a diagonal matrix whose diagonal entries form a vector $\mathbf{t} \in \mathbb{R}_+^k$, the function in (2.2) (as a function of the vector-argument $\mathbf{t}$) becomes the LT of a random vector in $\mathbb{R}_+^k$ consisting of the main diagonal elements of $\mathbf{X} \sim \mathcal{MG}_k(\alpha, \mathbf{A})$. This provides a link between MG distributions on $\mathbb{S}_k^+$, given by the LT (2.2) of a matrix-argument $\mathbf{T}$, and multivariate gamma distributions on $\mathbb{R}_+^k$, defined by the LT (2.2) of vector-argument $\mathbf{t}$, which is the main diagonal of the diagonal matrix $\mathbf{T}$. There is a substantial literature on multivariate gamma distributions with such a structure (see, e.g.,* [14]*;* [10]*;* [29]*;* [43]*; and references therein), most recently in connection with the so called permanental process, where these distributions provide their multivariate marginal distributions (see, e.g.,* [10]*). These processes are generalizations of the squared and centered Gaussian process, which arise in this setting when $\alpha = 1/2$. This is related to the fact that if $\alpha = 1/2$, the function in (2.2) with diagonal $\mathbf{T}$ is the LT of the random vector $(Y_1^2/2, \dots, Y_k^2/2)^\top$ where $(Y_1, \dots, Y_k)^\top$ has a (centered) multivariate normal distribution with the covariance matrix $\mathbf{A}$.*

The MG distributions have a desirable scaling invariance property, formulated below.

Proposition 2.1. *Let $\mathbf{A} \in \mathbb{S}_k^+$ and let $\mathbf{L}$ be an arbitrary $q \times k$ matrix of constants such that rank$(\mathbf{L}) = q \le k$. Then $\mathbf{L}\mathbf{X}\mathbf{L}^\top \sim \mathcal{MG}_q(\alpha, \mathbf{L}\mathbf{A}\mathbf{L}^\top)$ whenever $\mathbf{X} \sim \mathcal{MG}_k(\alpha, \mathbf{A})$, $\alpha > (k-1)/2$.*

For the proof of this or other results of this work, see the Appendix.

From the above result we have the following permutation invariance of the matrix gamma distributions. Let π be a permutation of $\{1, \dots, k\}$ that is also identified with its permutation matrix $(\delta_{i\pi(j)})_{i,j=1}^k$, where $\delta_{kj} = 1$, if $k = j$ and zero otherwise. Thus, we have $\pi^\top = \pi^{-1}$ and $\pi\mathbf{X}\pi^\top = (X_{\pi^{-1}(i),\pi^{-1}(j)})_{i,j=1}^k$

Corollary 2.2. *If $\mathbf{X} \sim \mathcal{MG}_k(\alpha, \mathbf{A})$ where $\alpha > (k-1)/2$, then, for a permutation matrix π, $\pi\mathbf{X}\pi^\top \sim \mathcal{MG}_k(\alpha, \pi\mathbf{A}\pi^\top)$. In particular, in the standard case $\mathbf{X} \sim \mathcal{MG}_k(\alpha)$ with $\alpha > (k-1)/2$, we have $\mathbf{X} \stackrel{d}{=} \pi\mathbf{X}\pi^\top$.*

Remark 6. *It should be noted that the family is closed under the above 'sandwiched' scaling but not under the one-sided scaling. Indeed, as shown in Proposition 2.8 below, we have the following stochastic representation of $\mathbf{X} \sim \mathcal{MG}_2(\alpha)$:*

$$\mathbf{X} \stackrel{d}{=} \begin{pmatrix} \Gamma & (\Gamma/2)^{1/2} Z \\ (\Gamma/2)^{1/2} Z & Z^2/2 + \tilde{\Gamma} \end{pmatrix}, \tag{2.6}$$

where the variables Γ and $\tilde{\Gamma}$ have standard gamma distributions with shape parameters $\alpha > 1/2$ and $\alpha - 1/2$, respectively, while the Z has standard normal distribution, with all three variables mutually independent. For a positive definite matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

we obtain

$$\mathbf{A}\mathbf{X} \stackrel{d}{=} \begin{pmatrix} \Gamma + (\Gamma/2)^{1/2} Z & (\Gamma/2)^{1/2} Z + Z^2/2 + \tilde{\Gamma} \\ \Gamma + 2(\Gamma/2)^{1/2} Z & (\Gamma/2)^{1/2} Z + Z^2 + 2\tilde{\Gamma} \end{pmatrix},$$

which clearly does not have MG distribution (is not even positive definite).

In the next result, we recall the distributional structure of the blocks of the random matrix $\mathbf{X} \sim \mathcal{MG}_k(\alpha, \mathbf{A})$. To formulate the result, we use the standard notation for matrix-variate normal distribution, see [15]. Namely, we write $\mathbf{X} \sim \mathcal{MN}_{p,r}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$ if the elements of the random matrix $\mathbf{X}$ are jointly normally distributed, the $p \times r$ matrix $\mathbf{M}$ is the matrix of the expectations of the entries, and $\mathbf{\Sigma} \otimes \mathbf{\Psi} = \text{cov}\left(\text{vec}\left(\mathbf{X}^\top\right)\right)$, where $\mathbf{\Sigma}$ and $\mathbf{\Psi}$ are $p \times p$ and $r \times r$ non-negative definite matrices. Such a random matrix can be conveniently represented as $\mathbf{X} \stackrel{d}{=} \mathbf{M} + \mathbf{\Sigma}^{1/2}\mathbf{Z}\mathbf{\Psi}^{1/2}$, where $\mathbf{Z}$ is a $p \times r$ random matrix with IID standard normal entries. Further, we also say that a $r \times r$ positive definite matrix $\mathbf{X}$ follows a matrix-variate generalized inverse Gaussian (MGIG) distribution, denoted by $\mathbf{X} \sim \mathcal{MGIG}_r(\lambda, \mathbf{\Phi}, \mathbf{\Psi})$, if its PDF is given by

$$f(\mathbf{X}) = \frac{2^{r\lambda}|\mathbf{\Phi}|^{-\lambda}}{B_\lambda\left(\frac{1}{4}\mathbf{\Psi}\mathbf{\Phi}\right)}|\mathbf{X}|^{\lambda-(r+1)/2}\text{etr}\left\{-\frac{1}{2}\left(\mathbf{\Phi}\mathbf{X}^{-1} + \mathbf{\Psi}\mathbf{X}\right)\right\},$$

where $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are symmetric non-negative definite matrices, $\lambda \in \mathbb{R}$, and $B_\lambda(\cdot)$ is the Type-2 Bessel function of Herz of matrix argument, see [18].

We note several basic distributional properties of the MGIG distributions that are relevant for this paper (for a more complete account see, for example, [7] and [26]). First, it is clear that $\mathcal{MGIG}_r(\lambda, \mathbf{0}, \mathbf{\Psi})$ coincides with $\mathcal{MG}_r(\lambda, 2\mathbf{\Psi}^{-1})$. Moreover, for $\mathbf{X} \sim \mathcal{MGIG}_r(\lambda, \mathbf{\Phi}, \mathbf{\Psi})$ we have

$$\mathbf{X}^{-1} \sim \mathcal{MGIG}_r(-\lambda, \mathbf{\Psi}, \mathbf{\Phi}), \tag{2.7}$$

$$\mathbf{A}\mathbf{X}\mathbf{A}^\top \sim \mathcal{MGIG}_r(\lambda, \mathbf{A}\mathbf{\Phi}\mathbf{A}^\top, {\mathbf{A}^\top}^{-1}\mathbf{\Psi}\mathbf{A}^{-1}), \tag{2.8}$$

where $\mathbf{A}$ is an invertible $r \times r$ scalar matrix (see, e.g., [3], [37]).

In what follows, for a block matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix},$$

with $\mathbf{A}_{11}$ ($\mathbf{A}_{22}$) invertible, the Schur complement of $\mathbf{A}_{11}$ ($\mathbf{A}_{22}$) is denoted by $\mathbf{A}_{22\cdot 1}$ ($\mathbf{A}_{11\cdot 2}$), i.e. $\mathbf{A}_{22\cdot 1} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$ ($\mathbf{A}_{11\cdot 2} = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$).

Proposition 2.3. *Consider the partition of $\mathbf{X} \sim \mathcal{MG}_k(\alpha, \mathbf{A})$ and its dispersion matrix $\mathbf{A} \in \mathbb{S}_k^+$, where $\alpha > (k-1)/2$, into the blocks*

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{X}_{12}^\top & \mathbf{X}_{22} \end{pmatrix} \quad \textit{and} \quad \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \tag{2.9}$$

with $\dim(\mathbf{X}_{11}) = \dim(\mathbf{A}_{11}) = r \times r$, $r = 1, \ldots, k-1$. *Then it holds that*

(i) $\mathbf{X}_{11} \sim \mathcal{MG}_r(\alpha, \mathbf{A}_{11})$,
(ii) $\mathbf{X}_{22} \sim \mathcal{MG}_{k-r}(\alpha, \mathbf{A}_{22})$,
(iii) $\mathbf{X}_{12}^\top|\mathbf{X}_{11} \sim \mathcal{MN}_{k-r,r}\left(\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{X}_{11}, \frac{1}{2}\mathbf{A}_{22\cdot 1} \otimes \mathbf{X}_{11}\right)$,
(iv) $\mathbf{X}_{22\cdot 1} \sim \mathcal{MG}_{k-r}\left(\alpha - \frac{r}{2}, \mathbf{A}_{22\cdot 1}\right)$,
(v) $\mathbf{X}_{11}|\mathbf{X}_{12} \sim \mathcal{MGIG}_r\left(\alpha - \frac{k-r}{2}, 2\mathbf{X}_{12}\mathbf{A}_{22\cdot 1}^{-1}\mathbf{X}_{21}, 2\mathbf{A}_{11\cdot 2}^{-1}\right)$.
(vi) $\mathbf{X}_{12}|\mathbf{X}_{22} \sim \mathcal{MN}_{r,k-r}\left(\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{X}_{22}, \frac{1}{2}\mathbf{A}_{11\cdot 2} \otimes \mathbf{X}_{22}\right)$,

Moreover, $(\mathbf{X}_{11}\ \mathbf{X}_{12})$ *is independent of* $\mathbf{X}_{22\cdot 1}$.

Remark 7. *The above result is partially formulated in* [16, p. 4] *without explicitly stating (ii), while part (v) is formulated in* [7, Theorem 1]. *However, since the proof of Theorem 3.3.9 in* [15] *and the proof of Theorem 1 in* [7], *which shows the complete result for the Wishart case, applies without change to the MG case (see* *also* [30] *for a more rigorous proof) the proof of Proposition 2.3 shall be omitted. We only point out that (vi) follows from (iii) in view of the permutation invariance given in Corollary 2.2 and applied to the permutation that swaps the blocks in* $\mathbf{X}$ *(and thus those in* $\mathbf{A}$ *as well).*

There are several distributional relations that follow directly from the above result, connected to the so-called Matsumoto-Yor property, see [7], [26], and [37], which discusses how this property relates to the above structural form of the MG distributions. Here, we limit ourselves to the following general relations between MGIG and MG distributions, which follow from Proposition 2.3 and the relations (2.7) - (2.8) by arguments that are summarized in the Appendix.

Corollary 2.4. *For positive integers* u, v, *let* $\mathbf{A} \in \mathbb{S}_u^+$, $\mathbf{B} \in \mathbb{S}_v^+$, $\mathbf{C}$ *be a* $u \times v$ *matrix,* $\alpha > (u+v-1)/2$, *and*

$$\begin{aligned} \mathbf{X} &\sim \mathcal{MGIG}_u\left(\frac{v}{2} - \alpha, 2\mathbf{A}, 2\mathbf{CBC}^\top\right), \\ \mathbf{Y} &\sim \mathcal{MGIG}_v\left(\alpha - \frac{u}{2}, 2\mathbf{C}^\top\mathbf{AC}, 2\mathbf{B}\right), \\ \mathbf{Z} &\sim \mathcal{MG}_v\left(\alpha - \frac{u}{2}, \mathbf{B}^{-1}\right), \end{aligned}$$

where $\mathbf{X}$, $\mathbf{Y}$, *and* $\mathbf{Z}$ *are mutually independent. Then, we have*

$$\mathbf{Y} \stackrel{d}{=} \mathbf{C}^\top\mathbf{XC} + \mathbf{Z}.$$

There are some specifications of the above result that deserve to be stated explicitly.

Remark 8. *If* $u = v$ *and* $\mathbf{C}$ *is invertible, then we have*

$$\mathbf{Y} \stackrel{d}{=} \mathbf{X} + \mathbf{Z},$$

where $\mathbf{Y} \sim \mathcal{MGIG}_u\left(\alpha - \frac{u}{2}, 2\mathbf{C}^\top\mathbf{AC}, 2\mathbf{B}\right)$, $\mathbf{X} \sim \mathcal{MGIG}_u\left(\frac{u}{2} - \alpha, 2\mathbf{C}^\top\mathbf{AC}, 2\mathbf{B}\right)$, *and* $\mathbf{Z} \sim \mathcal{MG}_u\left(\alpha - \frac{u}{2}, \mathbf{B}^{-1}\right)$. *In addition, if* $\mathbf{C} = \mathbf{I}$ *and* $\mathbf{A} = \mathbf{B}$, *then*

$$\mathbf{Y} \stackrel{d}{=} \mathbf{Y}^{-1} + \mathbf{Z},$$

where $\mathbf{Y} \sim \mathcal{MGIG}_u\left(\alpha - \frac{u}{2}, 2\mathbf{A}, 2\mathbf{A}\right)$ *and* $\mathbf{Z} \sim \mathcal{MG}_u\left(\alpha - \frac{u}{2}, \mathbf{A}^{-1}\right)$.

Corollary 2.5. *In the notation of Proposition 2.3, let* $\mathbf{X} \sim \mathcal{MG}_k(\alpha, \mathbf{A})$ *and define*

$$\tilde{\mathbf{X}} = \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{X}_{12}^\top & \mathbf{X}_{22\cdot 1} \end{pmatrix}. \tag{2.10}$$

Then the LT of $\tilde{\mathbf{X}}$ *is given by*

$$\psi_{\tilde{\mathbf{X}}}(\mathbf{T}) = |\mathbf{I}_k + \mathbf{AT}|^{-\alpha} \, |\mathbf{I}_{k-r} + \mathbf{A}_{22\cdot 1}\mathbf{T}_{22}|^{\alpha - r/2}.$$

Remark 9. *If* $\mathbf{X} \sim \mathcal{MG}_k\left(\frac{n}{2}, 2\mathbf{\Sigma}\right)$ *with* $\mathbf{\Sigma} > 0$, *then* $\mathbf{X}$ *has a* k*-variate Wishart distribution with* n *degrees of freedom and covariance parameter* $\mathbf{\Sigma}$, *denoted by* $\mathbf{X} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$. *We note that while the formal definition of MG distribution requires that* $n \geq k$, *it is well-known that for Wishart distributions this requirement is not necessary, and for* $n < k$ *we obtain singular Wishart distributions. This singular case is discussed further in Subsection 2.3.*

The MG distribution has several interesting stochastic representations. We present them below for the standard MG case, with proofs in the Appendix. Extensions to the general case are straightforward via the scaling property stated in Proposition 2.1.

Let $Z_\alpha = \delta {\Gamma_\alpha}^{1/2}$, where δ is a (symmetric) Rademacher variable (± 1 with equal probabilities), independent of the standard gamma variable Γ_α with shape parameter α. We refer to this variable and its distribution as *generalized symmetric Rayleigh*, since in the case $\alpha = 1$ it is the classical (symmetrized) Rayleigh distribution. We also note that the special case $\alpha = 1/2$ corresponds to normal distribution with mean zero and variance $1/2$. For $\alpha > 1/2$, consider a random, triangular 2×2 matrix

$$\mathbf{Z}_{2,\alpha} = \begin{pmatrix} Z_\alpha & 0 \\ Z_{1/2} & Z_{\alpha-1/2} \end{pmatrix} \tag{2.11}$$

with independent entries. The variable $\mathbf{Z}_{2,\alpha}$ and its distribution will be referred to as *generalized matrix Rayleigh* of dimension 2. Note that $\mathbf{Z}_{2,\alpha}\mathbf{Z}_{2,\alpha}^\top$ has standard MG distribution. We can generalize this to an arbitrary dimension k, by defining a generalized (triangular) matrix-variate Rayleigh (MR) distribution with parameter $\alpha > (k-1)/2$ through the recurrence relation

$$\mathbf{Z}_{k,\alpha} \stackrel{d}{=} \begin{pmatrix} \mathbf{Z}_{r,\alpha} & \mathbf{0} \\ \mathbf{Z}_0/\sqrt{2} & \mathbf{Z}_{k-r,\alpha-r/2} \end{pmatrix}, \tag{2.12}$$

where $r < k$ and $\mathbf{Z}_0$ has a standard $(k-r) \times r$ matrix-variate normal distribution, $\mathbf{Z}_0 \sim \mathcal{MN}_{k-r,r}\left(\mathbf{0}, \mathbf{I}_{k-r} \otimes \mathbf{I}_r\right)$. In this recurrence, we assume the independence of the blocks of the matrix on the right-hand-side of the relation. We shall write $\mathbf{Z} \sim \mathcal{MR}_k(\alpha)$ for such a random matrix, where $\alpha > (k-1)/2$. In the special case with $k = 1$ we set $\mathbf{Z}_{1,\alpha} \stackrel{d}{=} Z_\alpha$, where we have $\alpha > 0$.

The decomposition of a gamma distributed random matrix into the product of a triangular Rayleigh matrix and its transpose is well known, see [44] and Proposition 2.8 below. Before we turn to this result, we point out an interesting generalization of the discussed construction in the result below, with the proof in the Appendix.

Proposition 2.6. *For $\alpha, \beta > 0$, consider the following generalization of* (2.11)

$$\mathbf{Z} = \begin{pmatrix} Z_{\alpha+\beta} & 0 \\ Z_\alpha & Z_\beta \end{pmatrix}, \tag{2.13}$$

where the entries are independent variables distributed as generalized symmetric Rayleigh with the corresponding parameters. Then the following holds:

$$\mathbf{Z}\mathbf{Z}^\top = \begin{pmatrix} Z_{\alpha+\beta}^2 & Z_{\alpha+\beta}Z_\alpha \\ Z_{\alpha+\beta}Z_\alpha & Z_\alpha^2 + Z_\beta^2 \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} Z_\alpha^2 + Z_\beta^2 & Z_{\alpha+\beta}Z_\alpha \\ Z_{\alpha+\beta}Z_\alpha & Z_{\alpha+\beta}^2 \end{pmatrix}.$$

Corollary 2.7. *Let $\alpha < 1/2$ and*

$$\mathbf{Z} = \begin{pmatrix} Z_{1/2} & 0 \\ Z_\alpha & Z_{1/2-\alpha} \end{pmatrix}. \tag{2.14}$$

Then

$$\mathbf{Z}\mathbf{Z}^\top = \begin{pmatrix} Z_{1/2}^2 & Z_{1/2}Z_\alpha \\ Z_{1/2}Z_\alpha & Z_\alpha^2 + Z_{1/2-\alpha}^2 \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} Z_\alpha^2 + Z_{1/2-\alpha}^2 & Z_{1/2}Z_\alpha \\ Z_{1/2}Z_\alpha & Z_{1/2}^2 \end{pmatrix}.$$

The above results suggest a possibility of defining a 2×2 MG distribution for any $\alpha > 0$. Our work is aiming at such an extension, which is related to the result in Corollary 2.7 and presented throughout the paper. For now, we provide the following representations, valid for an arbitrary dimension k and standard MG case with $\alpha \geq k/2$, which easily follow from Proposition 2.3.

Proposition 2.8. *Let* $\mathbf{\Gamma} \sim \mathcal{MG}_k(\alpha)$ *where* $\alpha \geq \frac{k}{2}$*. Further, for* $r < k$*, let* $\mathbf{\Gamma}_0 \sim \mathcal{MG}_r(\alpha)$*,* $\mathbf{\Gamma}_1 \sim \mathcal{MG}_{k-r}\left(\alpha - \frac{k-r}{2}\right)$*, and* $\mathbf{Z}_0 \sim \mathcal{MN}_{k-r,r}\left(\mathbf{0}, \mathbf{I}_{k-r} \otimes \mathbf{I}_r\right)$*, all being mutually independent. Finally, set*

$$\mathbf{W} = \begin{pmatrix} \mathbf{\Gamma}_0^{1/2} \\ \mathbf{Z}_0/\sqrt{2} \end{pmatrix}, \ \mathbf{V} = \begin{pmatrix} \mathbf{0} \\ \mathbf{\Gamma}_1^{1/2} \end{pmatrix}.$$

Then, we have

$$\begin{aligned} \mathbf{\Gamma} &\stackrel{d}{=} \mathbf{W}\mathbf{W}^\top + \mathbf{V}\mathbf{V}^\top \\ &= \begin{pmatrix} \mathbf{\Gamma}_0 \\ \frac{1}{\sqrt{2}}\mathbf{Z}_0\mathbf{\Gamma}_0^{1/2} \end{pmatrix} \mathbf{\Gamma}_0^{-1} \left(\mathbf{\Gamma}_0 \ \frac{1}{\sqrt{2}}\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0^\top\right) + \begin{pmatrix} \mathbf{0} \\ \mathbf{\Gamma}_1^{1/2} \end{pmatrix} \left(\mathbf{0} \ \mathbf{\Gamma}_1^{1/2}\right) \\ &= \begin{pmatrix} \mathbf{\Gamma}_0 & \frac{1}{\sqrt{2}}\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0^\top \\ \frac{1}{\sqrt{2}}\mathbf{Z}_0\mathbf{\Gamma}_0^{1/2} & \frac{1}{2}\mathbf{Z}_0\mathbf{Z}_0^\top \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}_1 \end{pmatrix}. \end{aligned} \tag{2.15}$$

In addition, the following triangular representations hold:

$$\mathbf{\Gamma} \stackrel{d}{=} \mathbf{U}\mathbf{U}^\top \stackrel{d}{=} \mathbf{Z}\mathbf{Z}^\top, \tag{2.16}$$

where $\mathbf{Z} \sim \mathcal{MR}_k(\alpha)$ *and*

$$\mathbf{U} = \begin{pmatrix} \mathbf{\Gamma}_0^{1/2} & \mathbf{0} \\ \frac{1}{\sqrt{2}}\mathbf{Z}_0 & \mathbf{\Gamma}_1^{1/2} \end{pmatrix}.$$

Remark 10. *Due to the explicit nature of the representations and the independence of the entries of* $\mathbf{Z}$ *and* $\mathbf{U}$ *in* (2.16)*, one can conveniently use these relations to simulate random MG variates. Of the two representations, the triangular one involving* $\mathbf{Z}$ *is perhaps most suitable for this purpose as it does not involve any roots of random matrices. However, both representations require simulation of gamma and normal variables. See also Remark 18 in Subsection 3.1 for explicit representations that include the singular MG case as well.*

A simple consequence of the above stochastic representations is the result below concerning the expectation and the covariance of a non-singular MG distribution. The presented results can be deduces from the mathematically elegant results in [24]. The explicit results for the Wishart case, where the matrix normal moments have been utilized, are available in the literature, see, for example, [19]. Nevertheless, we present an independent derivation, which is useful for analogous results for the singular MG case. Let us note that the existing results in this direction may not be easily accessible by practitioners (see, e.g., [36], where the three dimensional case is presented as a new result, and also [25], where more general results have been obtained but, to the best of our knowledge, were not published as a journal article).

To formulate the results, we need the notion of the $k^2 \times k^2$ commutation matrix $\mathbf{K}_k$, consisting of k^2 of $k \times k$-blocks $\mathbf{S}_{ij}$ defined through their entries,

$$S_{ij;rs} = \begin{cases} 1; & i = s, j = r, \\ 0; & \text{otherwise.} \end{cases} \tag{2.17}$$

Proposition 2.9. *Let $\mathbf{X}$ be a $k \times k$ random matrix, such that $\mathbf{X} = \mathbf{X}^\top$, the entries on and above the diagonal are uncorrelated, and their variances are given by the matrix $\boldsymbol{\Sigma} = \left(\sigma_{ij}^2\right)_{i,j=1}^k$. Then, the covariance matrix of $\mathbf{X}$ (defined as the covariance of $\operatorname{vec} \mathbf{X}$, where vec is the vectorization operator on matrices) is given by*

$$\begin{aligned} \mathbf{C}_{\mathbf{X}} &= \operatorname{diag}\left(\operatorname{vec} \boldsymbol{\Sigma}\right) + \begin{pmatrix} \mathbf{0} & \sigma_{12}^2 \mathbf{S}_{12} & \sigma_{13}^2 \mathbf{S}_{13} & \dots & \sigma_{1k}^2 \mathbf{S}_{1k} \\ \sigma_{21}^2 \mathbf{S}_{21} & \mathbf{0} & \sigma_{23}^2 \mathbf{S}_{23} & \dots & \sigma_{2k}^2 \mathbf{S}_{2k} \\ \sigma_{31}^2 \mathbf{S}_{31} & \sigma_{32}^2 \mathbf{S}_{32} & \mathbf{0} & \dots & \sigma_{3k}^2 \mathbf{S}_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1}^2 \mathbf{S}_{k1} & \sigma_{k2}^2 \mathbf{S}_{k2} & \sigma_{k3}^2 \mathbf{S}_{k3} & \dots & \mathbf{0} \end{pmatrix} \\ &= \operatorname{diag}\left(\operatorname{vec}\left(\boldsymbol{\Sigma} - \operatorname{diag} \boldsymbol{\Sigma}\right)\right) + \left(\boldsymbol{\Sigma} \otimes \mathbf{1}_k\right) \circ \mathbf{K}_k. \end{aligned}$$

Here, $\mathbf{1}_k$ is $k \times k$ matrix of 1's, $\circ$ is the Hadamar product of matrices, and $\operatorname{diag}(\mathbf{V})$ is a diagonal matrix with the entries of $\mathbf{V}$ on the diagonal in the case when $\mathbf{V}$ is a vector, and the diagonal of $\mathbf{V}$ if the latter is a square matrix.

A straightforward proof of the result is given in the Appendix.

We note that the assumptions about non-correlated entries of the above result are satisfied by $\boldsymbol{\Gamma} \sim \mathcal{M}G_k(\alpha)$, which follows, for example, from (2.15). Moreover, we note that $\mathbb{V}ar(\Gamma_{ii}) = \alpha$ and $\mathbb{V}ar(\Gamma_{ij}) = \mathbb{V}ar(\sqrt{\Gamma/2}Z) = \alpha/2$. This yields the matrix $\boldsymbol{\Sigma}_\alpha = \alpha \left(1/2^{1-\delta_{ij}}\right)_{i,j=1}^k$ of variances of $\boldsymbol{\Gamma}$, leading to the following result.

Corollary 2.10. *Let $\boldsymbol{\Gamma} \sim \mathcal{M}G_k(\alpha)$. Then*

$$\begin{aligned} \mathbb{E}(\boldsymbol{\Gamma}) &= \alpha \, \mathbf{I}_k, \\ \mathbb{C}ov\left(\operatorname{vec} \boldsymbol{\Gamma}\right) &= \frac{\alpha}{2} \left(\mathbf{I}_{k^2} + \mathbf{K}_k\right). \end{aligned}$$

More generally, if $\boldsymbol{\Gamma} \sim \mathcal{M}G_k(\alpha, \mathbf{A})$, $\mathbf{A} \in \mathbb{S}_k^+$, then

$$\begin{aligned} \mathbb{E}(\boldsymbol{\Gamma}) &= \alpha \mathbf{A}, \\ \mathbb{C}ov\left(\operatorname{vec} \boldsymbol{\Gamma}\right) &= \frac{\alpha}{2} \left(\mathbf{A} \otimes \mathbf{A}\right) \left(\mathbf{I}_{k^2} + \mathbf{K}_k\right). \end{aligned}$$

The second part of the corollary follows from the standard properties of vectorization and Kronecker product,

$$\begin{aligned} \operatorname{vec}(\mathbf{ABC}) &= \left(\mathbf{C}^\top \otimes \mathbf{A}\right) \operatorname{vec} \mathbf{B}, \\ \left(\mathbf{A} \otimes \mathbf{B}\right) \left(\mathbf{C} \otimes \mathbf{D}\right) &= \mathbf{AC} \otimes \mathbf{BD}, \\ (\mathbf{A} \otimes \mathbf{B}) \mathbf{K}_k &= \mathbf{K}_k (\mathbf{B} \otimes \mathbf{A}). \end{aligned}$$

2.2. **matrix-variate Laplace distribution.** In view of Proposition 2.3, one can observe that the off-diagonal blocks of dimensions $r \times l$ and $l \times r$, where $l = k - r$, belong to the matrix gamma mean-covariance Gaussian mixtures, i.e. they have one the following two general structures:

$$\mathbf{X}_L = \mathbf{\Gamma M} + \mathbf{\Gamma}^{1/2}\mathbf{Z},$$
$$\mathbf{X}_R = \mathbf{M}^\top\mathbf{\Gamma} + \mathbf{Z}^\top\mathbf{\Gamma}^{1/2},$$

where $\mathbf{M}$ is an $r\times l$ non-random matrix, $\mathbf{Z} \sim \mathcal{MN}_{r,l}(\mathbf{0}, \mathbf{\Psi}\otimes\mathbf{\Sigma})$, and $\mathbf{\Gamma} \sim \mathcal{MG}_r(\alpha, \mathbf{A})$. The corresponding distributions constitute a special subclass of the matrix-variate generalized hyperbolic distributions, considered in [42]. In the obvious analogy to uni- and multivariate generalized asymmetric Laplace distributions discussed in [20], we call these distributions *matrix-variate generalized asymmetric Laplace* (MAL). The class has four non-superfluous matrix valued parameters $\mathbf{M}$, $\mathbf{A}$, $\mathbf{\Sigma}$, $\mathbf{\Psi}$ (identifiable up to a numerical scaling). We note that in the off-diagonal blocks of the matrix gamma distribution we have $\mathbf{\Psi} = \mathbf{I}_r$. In the block notation of a matrix valued gamma distribution, the distributions of $\mathbf{X}_{12}$ and $\mathbf{X}_{21}$ have the matrix parameters $\mathbf{\Sigma} = \mathbf{A}_{22\cdot 1}/2$, $\mathbf{M} = \mathbf{A}_{11}^{-1}\mathbf{A}_{12}$, and $\mathbf{A} = \mathbf{A}_{11}$.

In the case when $\mathbf{\Psi} = \mathbf{I}_r$, we denote these distributions as $\mathcal{MAL}_{r,l}(\alpha; \mathbf{A}, \mathbf{M}, \mathbf{\Sigma})$ and $\mathcal{MAL}_{l,r}^\top(\alpha; \mathbf{A}, \mathbf{M}, \mathbf{\Sigma})$, respectively. It is not our intention to fully study this class, but rather report properties that follow directly from those of the MG distributions. For a dedicated study of this class we refer to [21].

In the first property, we show some distributional invariance properties between the left-hand-side and right-hand-side multiplication of the matrix normal by matrix gamma. It was first observed in [21] that in the vector case, the gamma matrix multiplication is identical to scalar gamma multiplication. Namely, it was shown that mixing a Gaussian vector with a gamma matrix is equivalent to mixing it with a gamma scale as the following distributional identity shows:

$$\mathbf{\Gamma m} + \mathbf{\Gamma}^{\frac{1}{2}}\mathbf{Z}\sigma \stackrel{d}{=} \mathbf{m}\Gamma + \mathbf{\Sigma}^{\frac{1}{2}}\mathbf{Z}\sqrt{\Gamma},$$

where $\sigma > 0$, $\mathbf{m} = (m_1 \dots m_r)^\top$ is a non-random column, $\mathbf{Z}$ is a standard $r \times 1$ Gaussian vector, $\Gamma \in \mathcal{MG}_1(\alpha)$ is a standard gamma variable, $\mathbf{\Gamma} \in \mathcal{MG}_r(\alpha)$, and

$$\mathbf{\Sigma} = \frac{1}{2}\left\{\mathbf{m}^\top\mathbf{m}\mathbf{I}_r - \mathbf{m}\mathbf{m}^\top\right\} + \sigma^2\mathbf{I}_r.$$

As an interesting consequence of this result, we observe that while the left-hand-side is defined through a gamma matrix $\mathbf{\Gamma}$ that is not infinitely divisible and requires $\alpha > \frac{r-1}{2}$, the right-hand-side is defined for any $\alpha > 0$ and is infinitely divisible. This observation contributes to understanding why the convolution properties do hold for the off-diagonal vector components in the matrix but not for the matrix itself. This somewhat surprising result for random vectors admits a generalization to random matrices, as presented below with proofs in the Appendix.

Proposition 2.11. *The LT of* $\mathbf{X} \sim \mathcal{MAL}_{r,l}(\alpha; \mathbf{A}, \mathbf{M}, \mathbf{\Sigma})$, $\alpha > \frac{r-1}{2}$, *for each* $\mathbf{T}$ *such that*

$$\mathbf{I}_r + \frac{\mathbf{A}^{\frac{1}{2}}(\mathbf{T}\mathbf{M}^\top + \mathbf{M}\mathbf{T}^\top - \mathbf{T}\mathbf{\Sigma}\mathbf{T}^\top)\mathbf{A}^{\frac{1}{2}}}{2} \in \mathbb{S}_r^+ \tag{2.18}$$

is given by

$$\psi_{\mathbf{X}}(\mathbf{T}) = \left|\mathbf{I}_r + \mathbf{A}\frac{\mathbf{T}\mathbf{M}^\top + \mathbf{M}\mathbf{T}^\top - \mathbf{T}\mathbf{\Sigma}\mathbf{T}^\top}{2}\right|^{-\alpha}.$$

The ChF of $\mathbf{X} \sim \mathcal{MAL}_{r,l}(\alpha; \mathbf{A}, \mathbf{M}, \mathbf{\Sigma})$, $\alpha > \frac{r-1}{2}$, for any $r \times l$ matrix $\mathbf{T}$ is given by

$$\phi_{\mathbf{X}}(\mathbf{T}) = \left| \mathbf{I}_r - \mathbf{A} \frac{\iota \left(\mathbf{T}\mathbf{M}^\top + \mathbf{M}\mathbf{T}^\top \right) - \mathbf{T}\mathbf{\Sigma}\mathbf{T}^\top}{2} \right|^{-\alpha}.$$

The corresponding formulas for $\mathbf{X}^\top \sim \mathcal{MAL}^\top_{l,r}(\alpha; \mathbf{A}, \mathbf{M}, \mathbf{\Sigma})$ are obtained by interchanging the $\mathbf{T}$ and $\mathbf{T}^\top$ on the right-hand-sides above.

Remark 11. *Observe that the LT of the standard $\mathbf{X} \sim \mathcal{MAL}_{r,l}(\alpha; \mathbf{I}_r, \mathbf{0}, \mathbf{I}_l)$, which is well defined for $\mathbf{T}$ such that*

$$\mathbf{I}_r - \frac{\mathbf{T}\mathbf{T}^\top}{2} \in \mathbb{S}_r^+,$$

is given by

$$\psi_{\mathbf{X}}(\mathbf{T}) = \left| \mathbf{I}_r - \frac{\mathbf{T}\mathbf{T}^\top}{2} \right|^{-\alpha},$$

while the ChF of $\mathbf{X}$, defined for an arbitrary $\mathbf{T}$, is of the form

$$\phi_{\mathbf{X}}(\mathbf{T}) = \left| \mathbf{I}_r + \frac{\mathbf{T}\mathbf{T}^\top}{2} \right|^{-\alpha}.$$

In addition, the following stochastic representation holds:

$$\mathbf{X} \stackrel{d}{=} \mathbf{Z}_{r,\alpha} \mathbf{Z}_0^\top / \sqrt{2}, \tag{2.19}$$

where $\mathbf{Z}_{r,\alpha} \sim \mathcal{MR}_r(\alpha)$ and $\mathbf{Z}_0 \sim \mathcal{MN}_{l,r}(\mathbf{0}, \mathbf{I}_l \otimes \mathbf{I}_r)$, where these matrix variables are mutually independent. This represenation is a direct consequence of (2.16)*.*

Remark 12. *The representation* (2.19) *can be conveniently used to simulate matrix-variate Laplace random variables, as it involves only independent normal and gamma variables, readily available in standard packages.*

Proposition 2.12. *Let $\mathbf{\Gamma} \sim \mathcal{MG}_r(\alpha)$, $\mathbf{\Gamma}' \sim \mathcal{MG}_l(\alpha)$, and $\mathbf{Z} \sim \mathcal{MN}_{r,l}(\mathbf{0}, \mathbf{I}_r \otimes \mathbf{I}_l)$ and $\alpha > (\min(r,l) - 1)/2$. Then for each $r \times l$ matrix $\mathbf{M}$, $\mathbf{A} \in \mathbb{S}_r^+$, and $\mathbf{\Sigma} \in \mathbb{S}_l^+$, we have*

$$\mathbf{A}^{\frac{1}{2}} \mathbf{\Gamma} \mathbf{A}^{\frac{1}{2}} \mathbf{M} + \left(\mathbf{A}^{\frac{1}{2}} \mathbf{\Gamma} \mathbf{A}^{\frac{1}{2}} \right)^{\frac{1}{2}} \mathbf{Z} \mathbf{\Sigma}^{\frac{1}{2}} \stackrel{d}{=} \mathbf{M'}^\top \mathbf{A'}^{\frac{1}{2}} \mathbf{\Gamma}' \mathbf{A'}^{\frac{1}{2}} + \mathbf{\Sigma'}^{\frac{1}{2}} \mathbf{Z} \left(\mathbf{A'}^{\frac{1}{2}} \mathbf{\Gamma}' \mathbf{A'}^{\frac{1}{2}} \right)^{\frac{1}{2}},$$

where

$$\mathbf{A}' = 2\mathbf{\Sigma} + \mathbf{M}^\top \mathbf{A} \mathbf{M}, \;\; \mathbf{M}' = \mathbf{A'}^{-1} \mathbf{M}^\top \mathbf{A}, \;\; \mathbf{\Sigma}' = \frac{1}{2} \left\{ \mathbf{A} - \mathbf{A} \mathbf{M} \mathbf{A'}^{-1} \mathbf{M}^\top \mathbf{A} \right\}.$$

By combining the above results, we obtain an algebraic identity for the determinants that can be of use when $r \gg l$. The result below is a generalization of Theorem 18.1.1 of [17] and Lemma 5.1 of [21].

Corollary 2.13. *If $\mathbf{T}$ is an $r \times l$ matrix with complex entries then for $\mathbf{\Sigma} \in \mathbb{S}_l^+$ and an $r \times l$ real matrix $\mathbf{M}$, we have*

$$|\mathbf{A}'| \left| \mathbf{I}_r + \frac{\mathbf{M}\mathbf{T}^\top + \mathbf{T}\mathbf{M}^\top - \mathbf{T}\mathbf{\Sigma}\mathbf{T}^\top}{2} \right| = \left| \mathbf{A}' + \frac{\mathbf{A}'\mathbf{T}^\top\mathbf{M} + \mathbf{M}^\top\mathbf{T}\mathbf{A}' - \mathbf{A}'\mathbf{T}^\top\mathbf{\Sigma}'\mathbf{T}\mathbf{A}'}{2} \right|,$$

where

$$\mathbf{A}' = \mathbf{M}^\top \mathbf{M} + 2\boldsymbol{\Sigma},$$
$$\boldsymbol{\Sigma}' = \frac{1}{2}\left(\mathbf{I}_r - \mathbf{M}\mathbf{A}'^{-1}\mathbf{M}^\top\right).$$

The only part that needs a justification is the equality holding beyond $\mathbf{T}$ satisfying the conditions of Proposition 2.11. However, as functions of $\mathbf{T}$, both sides are multivariate polynomials that coincide on an open set. Therefore, they have to coincide across the entire domain of complex matrices $\mathbf{T}$, since the coefficients of the two polynomials must be the same on both sides.

For square Laplace matrices, we have the following generalization of the well-known representation of a Laplace distributed random variable as the difference of two independent gamma variables (see, e.g., [20]).

Proposition 2.14. *In the standard square case where* $\mathbf{X} \sim \mathcal{MAL}_{r,r}(\alpha; \mathbf{0}, \mathbf{I}_r)$ *we have the stochastic representation*

$$\mathbf{X} + \mathbf{X}^\top \stackrel{d}{=} \boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_2,$$

where $\boldsymbol{\Gamma}_i$*,* $i = 1, 2$*, are IID* $\mathcal{MG}_r(\alpha, \sqrt{2}\mathbf{I}_r)$ *distributed matrices.*

2.3. **Singular Wishart distribution.** In Section 3, we propose an extension of MG distributions that allows for an arbitrary parameter $\alpha > 0$ and involves the singular MG distributions. The special case of the new family is the class of the singular Wishart distributions, which is a well-known family. Reference [41] contains a number of valuable results on this subclass. Below we summarize the most important properties of these distributions. The following definition of the singular Wishart matrix may not be the most natural one, but it fits well with the extension of the concept of random gamma matrices to the case with an arbitrary positive α.

Definition 1. *Let* $k, r \in \mathbb{N}$ *with* $r < k$*. A* $k \times k$ *random matrix* $\mathbf{X}$ *is said to have a singular Wishart distribution with* r *degrees of freedom and dispersion matrix* $\boldsymbol{\Sigma} = \mathbf{A}/2$ *if the following stochastic representation holds:*

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{X}_{12}^\top & \mathbf{X}_{12}^\top \mathbf{X}_{11}^{-1} \mathbf{X}_{12} \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}.$$

Here, $\mathbf{X}_{11} \sim \mathcal{MG}_r(r/2, \mathbf{A}_{11}) = \mathcal{W}_r(r, \mathbf{A}_{11}/2)$ *and*

$$\mathbf{X}_{12}^\top | \mathbf{X}_{11} \sim \mathcal{MN}_{k-r,r}\left(\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{X}_{11}, \frac{1}{2}\mathbf{A}_{22\cdot 1} \otimes \mathbf{X}_{11}\right),$$

where $\dim(\mathbf{X}_{11}) = \dim(\mathbf{A}_{11}) = r \times r$*.*

The above definition suggests that the distributions of different partitions of a random matrix characterized by this distribution differ depending upon which part of the matrix they are located in. In other words, one would expect that this class of distributions is not closed on permutations of the coordinates. However, this is not the case for the subclass of the singular Wishart distributions. In the result below we let π be a permutation of $(1, \dots, k)$ so that $\pi(1), \dots, \pi(k)$ is a change of the order of the coordinates. In this notation, for a $k \times k$ matrix $\mathbf{X}$ we let $\mathbf{X}_\pi = (X_{\pi(i)\pi(j)})$.

Theorem 2.15. *Let* $k, n \in \mathbb{N}$ *and let* π *be a permutation of* $(1, \dots, k)$*. If* $\mathbf{X} \sim \mathcal{W}_k(n, \boldsymbol{\Sigma})$*, then* $\mathbf{X}_\pi \sim \mathcal{W}_k(n, \boldsymbol{\Sigma}_\pi)$*.*

The next result is well-known, see [23] and [13], and provides a general convolution property for Wishart matrices, which covers both singular and non-singular cases.

Theorem 2.16. *If the $k \times k$ random matrices $\mathbf{X}_1, \ldots, \mathbf{X}_N$ are all independent and $\mathbf{X}_i \sim \mathcal{W}_k(n_i, \mathbf{\Sigma})$, $i = 1, \ldots, N$, and $\mathbf{\Sigma} > 0$, then $\sum_{i=1}^N \mathbf{X}_i \sim \mathcal{W}_k(n, \mathbf{\Sigma})$, where $n = \sum_{i=1}^N n_i$.*

Remark 13. *Theorem 2.16 shows that the sum of independent Wishart random matrices with the same dispersion matrix is again a Wishart random matrix. Let us note that no assumption is imposed here on the degrees of freedom for these Wishart matrices. In particular, the sum of singular and non-singular Wishart matrices could be either singular or non-singular Wishart matrix. Moreover, the sum of singular Wishart matrices could become a non-singular Wishart matrix.*

3. A singular matrix-variate gamma distribution

As seen in the previous section, we have a restriction of the shape α to the values in the open interval $((k-1)/2, \infty)$, except for the singular Wishart special case that allows for $\alpha \in (0, (k-1)/2]$ with $\alpha = i/2$, $i = 1, \ldots, k-1$. Here, we present a natural and consistent extension of this class that incorporates any value of the shape parameter, including *all real values* in the interval $(0, (k-1)/2]$. As will be seen below, due to the singularity of this family of distributions, the manner of introducing a matrix scaling parameter is not entirely obvious. For this reason, we shall start with an extension of the standard MG case, where the dispersion parameter is the identity matrix. Subsequently, we discuss different alternatives for the matrix dispersion parameter.

Remark 14. *Let us note that the term "singular gamma" matrix-variate distribution has recently appeared in* [31]. *However, the singular matrix-variate distribution studied by the authors of* [31] *was, in fact, the singular Wishart distribution with shape parameter $\alpha = q/2$, as it was defined through the distribution of $\mathbf{X}\mathbf{X}^\top$ with $\mathbf{X}$ being a $p \times q$ matrix-variate normal with $p > q$, given by the PDF (1.7) in that paper.*

3.1. A standard singular matrix-variate gamma distribution.

Throughout this subsection, $r = \lceil 2\alpha \rceil$ denotes the rank of the random matrix $\mathbf{X}$ described in the definition below. We use the standard notation $\lceil \cdot \rceil$ to denote the smallest integer upper bound (the ceiling function).

Definition 2. *A $k \times k$ random matrix $\mathbf{X}$ is said to have a standard singular lower-right matrix gamma distribution with parameter $\alpha \in (0, (k-1)/2]$ if*

$$\mathbf{X} \stackrel{d}{=} \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{X}_{12}^\top & \mathbf{X}_{12}^\top \mathbf{X}_{11}^{-1} \mathbf{X}_{12} \end{pmatrix}, \tag{3.1}$$

where $\mathbf{X}_{11} \sim \mathcal{MG}_r(\alpha)$ and $\mathbf{X}_{12}^\top | \mathbf{X}_{11} \sim \mathcal{MN}_{k-r,r}\left(\mathbf{0}, \frac{1}{2}\mathbf{I}_{k-r} \otimes \mathbf{X}_{11}\right)$, where $r = \lceil 2\alpha \rceil$.

In agreement with our previous notation, we denote the above distributions by $\mathcal{MG}_k(\alpha)$. Notice that the structure of the matrix is not symmetric, as the lower-right $(k-r) \times (k-r)$ block has different distribution than the upper-left $r \times r$ block. If the roles of the blocks in the above definition are reversed, we obtain a *singular upper-left MG* distribution, which we shall denote by $\mathcal{MG}^k(\alpha)$.

We observe the following conditional distributions, which can be obtained by the same arguments as those in the non-singular case.

Remark 15. *We have the following singular analogs of the block conditional distributions:*

$$\begin{aligned}
\mathbf{X}_{11}|\mathbf{X}_{12} &\sim \mathcal{MGIG}_r\left(\alpha - \frac{k-r}{2}, 2\mathbf{X}_{12}\mathbf{X}_{12}^\top, 2\mathbf{I}_r\right),\\
\mathbf{X}_{11}^{-1}|\mathbf{X}_{21} &\sim \mathcal{MGIG}_r\left(\frac{k-r}{2} - \alpha, 2\mathbf{I}_r, 2\mathbf{X}_{21}^\top\mathbf{X}_{21}\right),\\
\mathbf{X}_{22}|\mathbf{X}_{21} &= \mathbf{X}_{21}\left(\mathbf{X}_{11}^{-1}|\mathbf{X}_{21}\right)\mathbf{X}_{21}^\top,
\end{aligned}$$

where $\mathbf{X}_{22} = \mathbf{X}_{12}^\top\mathbf{X}_{11}^{-1}\mathbf{X}_{12}$.

Remark 16. *Let us note that the above definition can be re-formulated with any* $r \le \lceil 2\alpha \rceil$*, and many of the following results could be extended to this case. However, any extension of singularity with* $r < \lceil 2\alpha \rceil$ *can be equivalently obtained through an additional singularity in the scaling matrix multiplication for the upper-left* $r \times r$ *corner. Moreover, if* $\alpha = r/2$*, where* r *is an integer less than* k*, then* $\mathbf{X} \sim \mathcal{MG}_k(\alpha)$ *defined via (3.1) has a singular Wishart distribution with* r *degrees of freedom and* $\mathbf{A} = \mathbf{I}_k/2$*, as described in Definition 1. For these reasons, we restrict attention to the largest possible value of* r*, i.e.* $r = \lceil 2\alpha \rceil$.

Remark 17. *Let us also note that (3.1) can be written in the following equivalent forms:*

(3.2)

$$\begin{pmatrix}\mathbf{X}_{11}^{\frac{1}{2}}\\ \mathbf{X}_{21}\mathbf{X}_{11}^{-\frac{1}{2}}\end{pmatrix}\left(\mathbf{X}_{11}^{\frac{1}{2}}\;\mathbf{X}_{11}^{-\frac{1}{2}}\mathbf{X}_{12}\right) = \begin{pmatrix}\mathbf{X}_{11}\\ \mathbf{X}_{21}\end{pmatrix}\left(\mathbf{I}_r\;\mathbf{X}_{11}^{-1}\mathbf{X}_{12}\right) = \begin{pmatrix}\mathbf{X}_{11}\\ \mathbf{X}_{21}\end{pmatrix}\mathbf{X}_{11}^{-1}\left(\mathbf{X}_{11}\;\mathbf{X}_{12}\right).$$

A direct application of the singular matrix-variate normal distributions (see [15], Definition 2.4.1) leads to the following result.

Proposition 3.1. *In the above notation, let* $\mathbf{X} \sim \mathcal{MG}_k(\alpha)$ *with* $\alpha \in (0, \frac{k-1}{2})$*. Then, conditionally on* $\mathbf{X}_{11} = \mathbf{\Gamma}_0$*, we have the singular matrix-variate normal representation:*

$$\mathbf{X}_{\cdot 1} \stackrel{def}{=} \begin{pmatrix}\mathbf{X}_{11}\\ \mathbf{X}_{21}\end{pmatrix} \stackrel{d}{=} \begin{pmatrix}\mathbf{0}\\ \mathbf{I}_{k-r}\end{pmatrix}\mathbf{Z}_0\mathbf{\Gamma}_0^{1/2}/\sqrt{2} + \begin{pmatrix}\mathbf{\Gamma}_0\\ \mathbf{0}\end{pmatrix},$$

where $\mathbf{Z}_0 \sim \mathcal{MN}_{k-r,r}(\mathbf{0}, \mathbf{I}_{k-r} \otimes \mathbf{I}_r)$*. Thus,* $(\mathbf{X}_{\cdot 1}|\mathbf{X}_{11} = \mathbf{\Gamma}_0) \sim \mathcal{MN}_{k,r}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Gamma}_0/2|k-r, r)$*, where*

$$\mathbf{M} = \begin{pmatrix}\mathbf{\Gamma}_0\\ \mathbf{0}\end{pmatrix},\ \mathbf{\Sigma} = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{k-r}\end{pmatrix}.$$

This singular matrix-variate normal distribution resides on the set of all $k \times r$ *matrices* $\mathbf{X}_{\cdot 1}$ *such that the top* $r \times r$ *block,* $\mathbf{X}_{11}$*, is constant equal to* $\mathbf{\Gamma}_0$ *and the lower* $(k-r) \times r$ *block,* $\mathbf{X}_{21}$*, is an arbitrary one. The density over this set of matrices, say* $\mathcal{M}_{\mathbf{\Gamma}_0}$*, is given by*

$$f_{\mathbf{X}_{\cdot 1}|\mathbf{X}_{11}}(\mathbf{X}_{\cdot 1}|\mathbf{\Gamma}_0) = (2\pi)^{r(k-r)}\,2^{r/2}|\mathbf{\Gamma}_0|^{-r/2}\mathrm{etr}\left\{(\mathbf{X}_{\cdot 1} - \mathbf{M})\mathbf{\Gamma}_0^{-1}(\mathbf{X}_{\cdot 1} - \mathbf{M})^\top\right\},\ \mathbf{X}_{\cdot 1} \in \mathcal{M}_{\mathbf{\Gamma}_0}.$$

The following result provides a fundamental representation of standard singular lower-right matrix-variate distribution, which is analogous to the well-known representation of standard singular Wishart distribution as the product $\mathbf{Z}\mathbf{Z}^\top$ involving a standard normal random matrix $\mathbf{Z}$. The proof can be found in the Appendix.

Proposition 3.2. *Let* $\mathbf{X} \sim \mathcal{MG}_k(\alpha)$ *with* $\alpha \in (0, \frac{k-1}{2}]$*, and set* $r = \lceil 2\alpha \rceil$*. Then* $\mathbf{X}$ *admits the following stochastic representation:*

$$\mathbf{X} \stackrel{d}{=} \mathbf{Z}\mathbf{Z}^\top = \begin{pmatrix} \mathbf{\Gamma}_0 & \frac{1}{\sqrt{2}}\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0^\top \\ \frac{1}{\sqrt{2}}\mathbf{Z}_0\mathbf{\Gamma}_0^{1/2} & \frac{1}{2}\mathbf{Z}_0\mathbf{Z}_0^\top \end{pmatrix} = \begin{pmatrix} \mathbf{\Gamma}_0 \\ \frac{1}{\sqrt{2}}\mathbf{Z}_0\mathbf{\Gamma}_0^{1/2} \end{pmatrix} \mathbf{\Gamma}_0^{-1} \left(\mathbf{\Gamma}_0 \; \frac{1}{\sqrt{2}}\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0^\top \right),$$

where

$$\mathbf{Z} = \begin{pmatrix} \mathbf{\Gamma}_0^{1/2} \\ \frac{\mathbf{Z}_0}{\sqrt{2}} \end{pmatrix}$$

is a $k \times r$ *stochastic matrix with mutually independent* $\mathbf{\Gamma}_0 \sim \mathcal{MG}_r(\alpha)$ *and*

$$\mathbf{Z}_0 \sim \mathcal{MN}_{k-r,r}(\mathbf{0}, \mathbf{I}_{k-r} \otimes \mathbf{I}_r).$$

Moreover, the rank of $\mathbf{X}$ *is equal to* $r = \lceil 2\alpha \rceil$ *and its (singular) density, which resides on all non-negative definite matrices of the form (3.1) with positive definite* $\mathbf{X}_{11}$*, is given by*

$$f_{\mathbf{X}}(\mathbf{X}) = f_{\mathbf{X}}(\mathbf{X}_{11}, \mathbf{X}_{12}) = \frac{|\mathbf{X}_{11}|^{\alpha-(k+1)/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)} \operatorname{etr}\{-\mathbf{X}\}$$

while the LT of $\mathbf{X}$ *is well-defined for each symmetric* $k \times k$ *matrix* $\mathbf{T}$ *for which*

$$\mathbf{I}_r + \mathbf{T}_{11} - \mathbf{T}_{12}\left(\mathbf{I}_{k-r} + \mathbf{T}_{22}\right)^{-1}\mathbf{T}_{21} \in \mathbb{S}_r^+$$

and takes the form

$$\psi_{\mathbf{X}}(\mathbf{T}) = |\mathbf{I}_k + \mathbf{T}|^{-\alpha} |\mathbf{I}_{k-r} + \mathbf{T}_{22}|^{\alpha - \frac{r}{2}}. \tag{3.3}$$

The ChF for a symmetric matrix $\mathbf{T}$ *has the form*

$$\psi_{\mathbf{X}}(\mathbf{T}) = |\mathbf{I}_k - \iota\mathbf{T}|^{-\alpha} |\mathbf{I}_{k-r} - \iota\mathbf{T}_{22}|^{\alpha - \frac{r}{2}}. \tag{3.4}$$

Having the representation given in Proposition 3.2, it is straightforward to obtain the formulas for the mean and the variance of the singular MG distribution, which are analogous to the ones for the non-singular case given in Corollary 2.10. To use Proposition 3.2, we need to show that in the singular case the entries in $\mathbf{\Gamma}$ are uncorrelated, which is rather straightforward. Indeed, the entries in $\mathbf{\Gamma}_0$ and $\mathbf{Z}_0\mathbf{Z}_0^\top$ (given in Proposition 3.2) are uncorrelated as these are a non-singular gamma and Wishart matrices, respectively. The ith and jth columns of $\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0^\top$, $i > j > r$, are made of uncorrelated variables, since they are obtained as an inner product of independent rows in $\mathbf{Z}_0$. The entries i and s in the lth column are uncorrelated, since

$$\begin{aligned} \mathbb{E}\left(\sum_{j=1}^k \Gamma_{ij}^{1/2} Z_{lj} \sum_{j=1}^k \Gamma_{sj}^{1/2} Z_{lj}\right) &= \mathbb{E}\left(\sum_{j=1}^k \Gamma_{ij}^{1/2} Z_{lj}^2 \Gamma_{sj}^{1/2}\right) \\ &= \mathbb{E}\left(\sum_{j=1}^k \Gamma_{ij}^{1/2} \Gamma_{sj}^{1/2}\right) \\ &= \mathbb{E}(\Gamma_{is}) \end{aligned}$$

and the latter is zero as reported in Corollary 2.10. The terms in $\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0^\top$ and $\mathbf{Z}_0\mathbf{Z}_0^\top$ are uncorrelated since

$$\mathbb{E}\left(\sum_{j=1}^{k}\Gamma_{ij}^{1/2}Z_{lj}\sum_{j=1}^{k}Z_{lj}^2\right)=\mathbb{E}\left(\sum_{j=1}^{k}\Gamma_{ij}^{1/2}Z_{lj}^3\right)=0.$$

Finally, the terms in $\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0^\top$ and the terms in $\mathbf{\Gamma}_0$ are uncorrelated due to the independence of $\mathbf{Z}_0$ from $\mathbf{\Gamma}_0$.

We note also that

$$\begin{aligned}\mathbb{E}\left(\left(\sum_{j=1}^{k}\Gamma_{ij}^{1/2}Z_{lj}\right)^2\right)&=\mathbb{E}\left(\sum_{j=1}^{k}\Gamma_{ij}^{1/2}\Gamma_{ij}^{1/2}\right)\\&=\mathbb{E}(\Gamma_{ii}),\end{aligned}$$

so that the matrix $\mathbf{\Sigma}$ of the variances of $\mathbf{\Gamma}\sim\mathcal{M}G_k(\alpha)$, $\alpha<(k-1)/2$, is given by

$$\mathbf{\Sigma}=\begin{pmatrix}\alpha\mathbf{H}_r & \frac{\alpha}{2}\mathbf{1}_{r,k-r}\\ \frac{\alpha}{2}\mathbf{1}_{r,k-r}^\top & \frac{r}{2}\mathbf{H}_{k-r}\end{pmatrix}=\alpha\mathbf{H}_k+\left(\frac{r}{2}-\alpha\right)\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{H}_{k-r}\end{pmatrix},$$

where $\mathbf{H}_s=\left(1/2^{1-\delta_{ij}}\right)_{i,j=1}^{s}$, $s\in\mathbb{N}$. Thus, using the linearity in Proposition 3.2 with respect to $\mathbf{\Sigma}$, we obtain the following formulas for the mean and the covariances.

Proposition 3.3. *Let* $\mathbf{\Gamma}\sim\mathcal{M}G_k(\alpha)$, $\alpha<(k-1)/2$. *Then,*

$$\begin{aligned}\mathbb{E}(\mathbf{\Gamma})&=\begin{pmatrix}\alpha\,\mathbf{I}_r & \mathbf{0}\\ \mathbf{0} & \frac{r}{2}\,\mathbf{I}_{k-r}\end{pmatrix},\\ \mathbb{C}ov\,(\mathrm{vec}\,\mathbf{\Gamma})&=\frac{\alpha}{2}\left(\mathbf{I}_{k^2}+\mathbf{K}_k\right)+\\&\quad+\left(\frac{r}{2}-\alpha\right)\left(\mathrm{diag}\left(\mathrm{vec}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{H}_{k-r}\end{pmatrix}\right)+\frac{1}{2}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_{k-r}-\mathbf{I}_{k-r}\end{pmatrix}\right).\end{aligned}$$

We now extend the definition of the random Rayleigh matrix $\mathbf{Z}_{k,\alpha}\sim\mathcal{MR}_k(\alpha)$ in (2.11) to an arbitrary α using the recurrence in (2.12) and assuming that for a negative α the matrix (variable) vanishes, i.e. becomes a matrix of zeros. Below we provide a representation of singular MG variables in terms of these extended MR distributions, which is analogous to the one that holds in the non-singular case. This representation, which follows directly from the above result and Proposition 2.8, provides a natural method of simulating singular MG random matrices.

Corollary 3.4. *Let* $\mathbf{Z}_{k,\alpha}\sim\mathcal{MR}_k(\alpha)$ *where* $\alpha>0$. *Then* $\mathbf{Z}_\alpha\mathbf{Z}_\alpha^\top\sim\mathcal{MG}_k(\alpha)$.

Remark 18. *Below we provide two explicit triangular representations of an arbitrary (singular or not)* $\mathbf{X}\sim\mathcal{MG}_k(\alpha)$, *which are useful for simulation of these random matrices. First, for any* $\alpha>0$ *and* $i\in\{1,2,\dots,k-1\}$, *we define* $\alpha_i=\left(\alpha-\frac{i-1}{2}\right)^+$, *where* x^+ *is the positive part of* $x\in\mathbb{R}$. *In particular, we have* $\alpha_1=\alpha$. *We also define a binary* $\{\delta_i\}$ *where* $\delta_i=1$ *whenever* $\alpha>\frac{i-1}{2}$ *(and zero otherwise). Further, we let* $Z_{\alpha_i}\sim\mathcal{MR}_1(\alpha_i)$ *be one-dimensional generalized Rayleigh variables (which reduce to zero when* $\alpha_i=0$*) and we let* Γ_{α_i} *be one-dimensional standard gamma variables, with shape parameters given by* α_i *(which also become zero when* $\alpha_i=0$*). Finally, we let* $\{Z_{i,j}\}$ *be IID standard normal variables and we* *let $\{Z_{1/2}^{i,j}\}$ be IID copies of generalized symmetric Rayleigh variables with shape parameter 1/2. Under this notation, we define two $k\times k$ triangular random matrices as follows*

$$\mathbf{Z}_\alpha = \begin{pmatrix} Z_{k,\alpha_1} & 0 & 0 & 0 & 0 & \dots & 0 & 0\\ Z_{\frac12}^{2,1} & Z_{\alpha_2} & 0 & 0 & 0 & \dots & 0 & 0\\ Z_{\frac12}^{3,1} & \delta_2 Z_{\frac12}^{3,2} & Z_{\alpha_3} & 0 & 0 & \dots & 0 & 0\\ Z_{\frac12}^{4,1} & \delta_2 Z_{\frac12}^{4,2} & \delta_3 Z_{\frac12}^{4,3} & Z_{\alpha_4} & 0 & \dots & 0 & 0\\ Z_{\frac12}^{5,1} & \delta_2 Z_{\frac12}^{5,2} & \delta_3 Z_{\frac12}^{5,3} & \delta_4 Z_{\frac12}^{5,4} & Z_{\alpha_5} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ Z_{\frac12}^{k-1,1} & \delta_2 Z_{\frac12}^{k-1,2} & \delta_3 Z_{\frac12}^{k-1,3} & \delta_4 Z_{\frac12}^{k-1,4} & \delta_5 Z_{\frac12}^{k-1,5} & \dots & Z_{\alpha_{k-1}} & 0\\ Z_{\frac12}^{k,1} & \delta_2 Z_{\frac12}^{k,2} & \delta_3 Z_{\frac12}^{k,3} & \delta_4 Z_{\frac12}^{k,4} & \delta_5 Z_{\frac12}^{k,5} & \dots & \delta_{k-1} Z_{\frac12}^{k,k-1} & Z_{\alpha_k} \end{pmatrix}, \tag{3.5}$$

$$\mathbf{U}_{k,\alpha} = \begin{pmatrix} \Gamma_{\alpha_1}^{1/2} & 0 & 0 & 0 & 0 & \dots & 0 & 0\\ \frac{Z_{2,1}}{\sqrt2} & \Gamma_{\alpha_2}^{1/2} & 0 & 0 & 0 & \dots & 0 & 0\\ \frac{Z_{3,1}}{\sqrt2} & \delta_2\frac{Z_{3,2}}{\sqrt2} & \Gamma_{\alpha_3}^{1/2} & 0 & 0 & \dots & 0 & 0\\ \frac{Z_{4,1}}{\sqrt2} & \delta_2\frac{Z_{4,2}}{\sqrt2} & \delta_3\frac{Z_{4,3}}{\sqrt2} & \Gamma_{\alpha_4}^{1/2} & 0 & \dots & 0 & 0\\ \frac{Z_{5,1}}{\sqrt2} & \delta_2\frac{Z_{5,2}}{\sqrt2} & \delta_3\frac{Z_{5,3}}{\sqrt2} & \delta_4\frac{Z_{5,4}}{\sqrt2} & \Gamma_{\alpha_5}^{1/2} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \frac{Z_{k-1,1}}{\sqrt2} & \delta_2\frac{Z_{k-1,2}}{\sqrt2} & \delta_3\frac{Z_{k-1,3}}{\sqrt2} & \delta_4\frac{Z_{k-1,4}}{\sqrt2} & \delta_5\frac{Z_{k-1,5}}{\sqrt2} & \dots & \Gamma_{\alpha_{k-1}}^{1/2} & 0\\ \frac{Z_{k,1}}{\sqrt2} & \delta_2\frac{Z_{k,2}}{\sqrt2} & \delta_3\frac{Z_{k,3}}{\sqrt2} & \delta_4\frac{Z_{k,4}}{\sqrt2} & \delta_5\frac{Z_{k,5}}{\sqrt2} & \dots & \delta_{k-1}\frac{Z_{k,k-1}}{\sqrt2} & \Gamma_{\alpha_k}^{1/2} \end{pmatrix}, \tag{3.6}$$

where all the random entries in the above matrices are mutually independent. Then, we have the following stochastic representation of $\mathbf{X}\sim\mathcal{MG}_k(\alpha)$

$$\mathbf{X} \stackrel{d}{=} \mathbf{Z}_{k,\alpha}\mathbf{Z}_{k,\alpha}^\top \stackrel{d}{=} \mathbf{U}_{k,\alpha}\mathbf{U}_{k,\alpha}^\top.$$

We also have the following interesting representation of MG random matrices, which can be easily deduced from the above remark.

Theorem 3.5. *For each $\alpha>0$ and $k\in\mathbb{N}$, the following stochastic representation of $\mathbf{X}\sim\mathcal{MG}_k(\alpha)$ holds:*

$$\mathbf{X} \stackrel{d}{=} \mathbf{\Gamma}_\alpha + \mathbf{\Gamma}_\alpha^{1/2}\mathbf{Z}^\top + \mathbf{Z}\mathbf{\Gamma}_\alpha^{1/2} + \mathbf{Z}\mathbf{Z}^\top,$$

where $\mathbf{\Gamma}_\alpha$ and $\mathbf{Z}$ are independent, $\mathbf{\Gamma}_\alpha$ is a $k\times k$ random diagonal matrix with diagonal entries of the form Γ_{α_i}, $i=1,\dots,k$ (which are standard gamma variables with shape parameters α_i whenever $\alpha_i=\left(\alpha-\frac{i-1}{2}\right)^+>0$ and zeroes otherwise), and $\mathbf{Z}$ is a random triangular square matrix with zeros on and above the main diagonal, zeros below the main diagonal in the i-column if $\alpha_i=0$, and IID $\mathcal{N}(0,1/2)$ variables below the main diagonal in all other columns.

It follows from the results of [41] and the form of the LT given in (3.4), that the singular Wishart distribution is a scaled singular MG distribution, as reported next.

Corollary 3.6. *If $r = 2\alpha \le k-1$ is a positive integer and $\mathbf{X} \sim \mathcal{MG}_k(\frac{r}{2})$, then $\mathbf{X}/2$ has the singular Wishart distribution $\mathcal{W}_k(r, \mathbf{I}_k)$.*

Since for a singular MG random matrix $\mathbf{X} \sim \mathcal{M}G_k(\alpha)$ with $\alpha \le \frac{k-1}{2}$ the lower-right $(k-r)\times(k-r)$ block $\mathbf{X}_{22}$ of $\mathbf{X}$ is functionally dependent on the remaining blocks of $\mathbf{X}$, the distribution of $\mathbf{X}$ is uniquely determined by the distribution of

$$\mathbf{X}_0 = \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{X}_{12}^\top & \mathbf{0} \end{pmatrix}.$$

Thus, the LT of $\mathbf{X}_0$ uniquely defines the distribution of $\mathbf{X}$. If $\mathbf{T}$ is a symmetric matrix and $\mathbf{T}_0$ is the same as $\mathbf{T}$ but with the lower-right corner set to zero in the same way as in $\mathbf{X}_0$, then

$$\mathbb{E}\left[\operatorname{etr}\{-\mathbf{T}_0\mathbf{X}\}\right] = \mathbb{E}\left[\operatorname{etr}\{-\mathbf{T}\mathbf{X}_0\}\right].$$

Consequently, the unrestricted LT of $\mathbf{X}_0$ coincides with the restricted LT of $\mathbf{X}$. This leads to the following result.

Corollary 3.7. *We note the following restricted LT of $\mathbf{X} \sim \mathcal{MG}_k(\alpha)$ with $\alpha > 0$*

$$\psi_{\mathbf{X}}(\mathbf{T}_0) = \psi_{\mathbf{X}_0}(\mathbf{T}) = |\mathbf{I}_k + \mathbf{A}\mathbf{T}_0|^{-\alpha}, \tag{3.7}$$

where

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{12}^\top & \mathbf{T}_{22} \end{pmatrix} \tag{3.8}$$

with a symmetric $\mathbf{T}_{11}$ of dimension $r \times r$ and where $\mathbf{T}_0$ is obtained from $\mathbf{T}$ by setting $\mathbf{T}_{22} = \mathbf{0}$. Since the lower-right block $\mathbf{X}_{22}$ of $\mathbf{X}$ is functionally dependent on the remaining blocks $\mathbf{X}_{11}$ and $\mathbf{X}_{12}$ (that also define the entire distribution), this restricted LT of $\mathbf{X}$ uniquely defines the distribution.

3.2. **A scaled singular matrix-variate gamma distribution.** As we have seen in Section 2, the non-negative definite matrix dispersion parameter $\mathbf{A}$ of a non-singular MG distribution was introduced by simply scaling both sides of the standard MG $\mathbf{X}$ by the square root of $\mathbf{A}$. According to Proposition 2.1, a more general matrix scaling parameter $\mathbf{L}$ yields $\mathbf{L}\mathbf{X}\mathbf{L}^\top$, which distributionally depends only on $\mathbf{L}\mathbf{A}\mathbf{L}^\top$ and thus results in another non-negative definite dispersion parameter. However, as will be seen below, for the singular case of $\mathbf{X} \sim \mathcal{MG}_k(\alpha)$ with $\alpha \le (k-1)/2$, the distribution of $\mathbf{L}\mathbf{X}\mathbf{L}^\top$ does not depend on $\mathbf{L}$ only through $\mathbf{L}\mathbf{L}^\top$. For this reason, we propose the following general definition of the scaled singular MG distribution.

Definition 3. *Let $\mathbf{X}$ be a standard singular $k \times k$ MG variable with parameter $\alpha \in (0, (k-1)/2]$ and let $\mathbf{L}$ be a $k \times k$ matrix. Then, $\mathbf{Y} \overset{d}{=} \mathbf{L}\mathbf{X}\mathbf{L}^\top$ is said to have an $\mathbf{L}$-scaled singular MG distribution, denoted by $\mathcal{SMG}_k(\alpha, \mathbf{L})$.*

By Corollary 3.6 and Corollary 3.4 of [41], it follows that the singular Wishart family is a sub-class of the scaled singular MG distributions, as shown below.

Corollary 3.8. *If $r = 2\alpha \le k-1$ is an integer, then the $\mathcal{SMG}_k\left(\frac{r}{2}, \sqrt{2}\mathbf{A}^{1/2}\right)$ distribution coincides with the singular Wishart distribution $\mathcal{W}_k(r, \mathbf{A})$.*

Below we obtain the exact form of the (singular) PDF of the scaled MG distributions for two particular forms of $\mathbf{L}$. But first, we describe the set on which the singular MG distribution resides on. The proof of the following result can be found in the Appendix.

Proposition 3.9. *Let $\mathbf{X} \sim \mathcal{SMG}_k(\alpha, \mathbf{L})$ with $\mathbf{L}$ of full rank, $\alpha \in (0,(k-1)/2]$, and $r = \lceil 2\alpha \rceil$. In the notation of Proposition 3.2 the following stochastic representation holds*

$$\mathbf{X} \stackrel{d}{=} \mathbf{L} \begin{pmatrix} \boldsymbol{\Gamma}_0 \\ \frac{1}{\sqrt{2}}\mathbf{Z}_0\boldsymbol{\Gamma}_0^{1/2} \end{pmatrix} \boldsymbol{\Gamma}_0^{-1} \left(\boldsymbol{\Gamma}_0 \ \frac{1}{\sqrt{2}}\boldsymbol{\Gamma}_0^{1/2}\mathbf{Z}_0^\top \right) \mathbf{L}^\top.$$

Moreover, $\mathbf{X}$ has the form (3.1) and its rank is equal to $\lceil 2\alpha \rceil$.

The following result concerning the LT of $\mathbf{X} \sim \mathcal{SMG}_k(\alpha, \mathbf{L})$ follows directly from Proposition 3.2, and its short proof can be found in the Appendix.

Proposition 3.10. *Let $\mathbf{X} \sim \mathcal{SMG}_k(\alpha, \mathbf{L})$ with a $k \times k$ matrix $\mathbf{L}$ and $\alpha \le (k-1)/2$. Then the LT of $\mathbf{X}$, evaluated at a symmetric $\mathbf{T}$ satisfying*

$$\mathbf{I}_r + \left(\mathbf{L}^\top\mathbf{T}\mathbf{L}\right)_{11} - \left(\mathbf{L}^\top\mathbf{T}\mathbf{L}\right)_{12}\left(\mathbf{I}_{k-r} + \left(\mathbf{L}^\top\mathbf{T}\mathbf{L}\right)_{22}\right)^{-1}\left(\mathbf{L}^\top\mathbf{T}\mathbf{L}\right)_{21} \in \mathbb{S}_r^+, \tag{3.9}$$

is given by

$$\psi_{\mathbf{X}}(\mathbf{T}) = |\mathbf{I}_k + \mathbf{L}\mathbf{L}^\top\mathbf{T}|^{-\alpha}\left|\mathbf{I}_{k-r} + \left(\mathbf{L}^\top\mathbf{T}\mathbf{L}\right)_{22}\right|^{\alpha-\frac{r}{2}}, \tag{3.10}$$

where $r = \lceil 2\alpha \rceil$ and $\mathbf{C}_{22}$ stands for the lower-right, $(k-r)\times(k-r)$ corner of a $k \times k$ matrix $\mathbf{C}$.

Although one can derive further properties of the scaled singular MG distribution for any general $\mathbf{L}$, we shall focus on two specific cases where $\mathbf{L}$ is related to a non-negative definite matrix dispersion parameter $\mathbf{A}$. In the first case, we take $\mathbf{L} = \mathbf{A}^{1/2}$ while in the second one we take $\mathbf{L} = \mathbf{A}_{2\cdot 1}$, where

$$\mathbf{A}_{2\cdot 1} = \begin{pmatrix} \mathbf{A}_{11}^{1/2} & \mathbf{0} \\ \mathbf{A}_{21}\mathbf{A}_{11}^{-1/2} & \mathbf{A}_{22\cdot 1}^{1/2} \end{pmatrix}. \tag{3.11}$$

Observe that in both cases we have $\mathbf{L}\mathbf{L}^\top = \mathbf{A}$. However, despite having the same dispersion, the two subclasses of the scaled singular MG distributions are essentially different. The following simple example in two dimensions illustrates the difference.

Example 1. *The standard singular MG variable $\mathbf{X}$ in two dimensions can be represented as*

$$\mathbf{X} \stackrel{d}{=} \begin{pmatrix} \Gamma & \sqrt{\Gamma/2}Z \\ \sqrt{\Gamma/2}Z & Z^2/2 \end{pmatrix},$$

where Z is standard normal and Γ is standard gamma with shape parameter $\alpha \le 1/2$. Consider the dispersion matrix $\mathbf{A}$ and two scaling matrices, $\mathbf{L}_1$ and $\mathbf{L}_2$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \ \mathbf{L}_1 = \mathbf{A}^{1/2} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \ \mathbf{L}_2 = \mathbf{A}_{2\cdot 1} = \begin{pmatrix} \sqrt{2} & 0 \\ 3/\sqrt{2} & 1/2 \end{pmatrix},$$

for which we have $\mathbf{L}_i\mathbf{L}_i^\top = \mathbf{A}$, $i = 1, 2$. Then, we have

$$\mathbf{L}_1\mathbf{X}\mathbf{L}_1^\top = \begin{pmatrix} (\sqrt{\Gamma} + Z/\sqrt{2})^2 & \Gamma + Z^2 + 3\sqrt{\Gamma/2}Z \\ \Gamma + Z^2 + 3\sqrt{\Gamma/2}Z & (\sqrt{\Gamma} + \sqrt{2}Z)^2 \end{pmatrix},$$

$$\mathbf{L}_2\mathbf{X}\mathbf{L}_2^\top = \begin{pmatrix} 2\Gamma & 3\Gamma + \sqrt{\Gamma}Z/2 \\ 3\Gamma + \sqrt{\Gamma}Z/2 & (3\sqrt{\Gamma} + Z/2)^2/2 \end{pmatrix}.$$

If $\alpha = 1/2$, so that $\mathbf{X}$ is singular Wishart $\mathcal{W}_2(1, \mathbf{I}_2/2)$, then, by Corollary 3.3 of [41], *in both cases we get one and the same singular Wishart distribution $\mathcal{W}_2(1, \mathbf{A}/2)$, although this is not easily seen by comparing the above matrices. However, when $0 < \alpha < 1/2$, the two distributions are essentially different. This can be seen by comparing the variables in the corresponding upper-left corners. Indeed, routine albeit tedious calculations show that the LT of the variable $(\sqrt{\Gamma} + Z/\sqrt{2})^2$ is of the form*

$$\psi(t) = \frac{1}{\sqrt{1+t}} \left(\frac{1+t}{1+2t} \right)^{\alpha}, \ \ t > 0,$$

which is not the same as the LT $(1+2t)^{-\alpha}$ of the gamma variable 2Γ (unless $\alpha = 1/2$ and $\mathbf{X}$ is singular Wishart).

3.3. **The scaling $\mathbf{L} = \mathbf{A}^{1/2}$.** The (singular) density and the LT of the singular MG distributions are discussed in the next two results. Rather technical proof of the result below can be found in the Appendix.

Theorem 3.11. *Let $\mathbf{X} \sim \mathcal{SMG}_k(\alpha, \mathbf{A}^{1/2})$, where $\alpha \le (k-1)/2$ and $\mathbf{X}$, $\mathbf{A}$ are partitioned as in (3.1). Then the support of the PDF of $\mathbf{X}$ is the subset of non-negative definite $k \times k$ matrices $\mathbf{X}$ of rank $r = \lceil 2\alpha \rceil$ having the form*

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{X}_{12}^\top & \mathbf{X}_{12}^\top \mathbf{X}_{11}^{-1} \mathbf{X}_{12} \end{pmatrix},$$

where $\dim(\mathbf{X}_{11}) = r \times r$. Further, the PDF is given by

$$f_{\mathbf{X}}(\mathbf{X}_{11}, \mathbf{X}_{12}) = \frac{|\mathbf{X}_{11}|^{(r-k-1)/2-\alpha}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)|\mathbf{A}|^{r/2}} |\mathbf{B}_{11}\mathbf{X}_{11} + \mathbf{B}_{12}\mathbf{X}_{12}^\top/\sqrt{2}|^{2\alpha - r} \mathrm{etr}\left\{-\mathbf{A}^{-1}\mathbf{X}\right\},$$

where $\mathbf{B} = \mathbf{A}^{-1/2}$ and its blocks of respective sizes $r \times r$, $r \times (k-r)$, $(k-r) \times r$, and $(k-r) \times (k-r)$ are denoted by $\mathbf{B}_{11}$, $\mathbf{B}_{12}$, $\mathbf{B}_{21}$, $\mathbf{B}_{22}$.

In an important special case where $\mathbf{A}_{12} = \mathbf{0}$ we have a substantial simplification of the PDF, presented below.

Corollary 3.12. *In the special case where $\mathbf{A}_{12} = \mathbf{0}$ we obtain the following simple form of the PDF of $\mathbf{X} \sim \mathcal{SMG}_k(\alpha, \mathbf{A}^{1/2})$:*

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{X}_{11}, \mathbf{X}_{12}) &= \frac{\pi^{r(r-k)/2}}{\Gamma_r(\alpha)|\mathbf{A}_{11}|^{\alpha}|\mathbf{A}_{22}|^{r/2}} |\mathbf{X}_{11}|^{\alpha-(k+1)/2} \mathrm{etr}\left\{-\mathbf{A}^{-1}\mathbf{X}\right\} \\ &= \frac{\pi^{r(r-k)/2}}{\Gamma_r(\alpha)|\mathbf{A}|^{\alpha}|\mathbf{A}_{22}|^{r/2-\alpha}} |\mathbf{X}_{11}|^{\alpha-(k+1)/2} \mathrm{etr}\left\{-\mathbf{A}^{-1}\mathbf{X}\right\}. \end{aligned} \tag{3.12}$$

The formula for the LT given in the result below follows directly from Proposition 3.10 with $\mathbf{L} = \mathbf{A}^{1/2}$.

Theorem 3.13. *Under the setting and assumptions of Theorem 3.11, the LT of $\mathbf{X}$ is given by*

$$\psi_{\mathbf{X}}(\mathbf{T}) = |\mathbf{I}_k + \mathbf{A}\mathbf{T}|^{-\alpha} \left| \mathbf{I}_{k-r} + \left(\mathbf{A}^{1/2}\mathbf{T}\mathbf{A}^{1/2} \right)_{22} \right|^{\alpha - \frac{r}{2}}, \tag{3.13}$$

where $r = \lceil 2\alpha \rceil$ and $\mathbf{C}_{22}$ stands for the lower-right, $(k-r) \times (k-r)$ corner of a $k \times k$ matrix $\mathbf{C}$.

The above result can be utilized to obtain the marginal distributions of the diagonal blocks. The result below, whose proof can be found in the Appendix, provides the details.

Theorem 3.14. *Under the setting and assumptions of Theorem 3.11, let* $\mathbf{X}_{11}$*,* $\mathbf{X}_{12} = \mathbf{X}_{21}$*, and* $\mathbf{X}_{22}$ *be the standard partition of* $\mathbf{X}$ *into the blocks, with* $\mathbf{X}_{11}$ *being an* $r \times r$ *matrix where* $r = \lceil 2\alpha \rceil$*. Then, we have the following LTs of the blocks*

$$\psi_{\mathbf{X}_{11}}(\mathbf{T}_{11}) = |\mathbf{I}_r + \mathbf{A}_{11}\mathbf{T}_{11}|^{-\alpha} \, |\mathbf{I}_r + \mathbf{B}_{12}\mathbf{B}_{21}\mathbf{T}_{11}|^{\alpha - \frac{r}{2}},$$
$$\psi_{\mathbf{X}_{22}}(\mathbf{T}_{22}) = |\mathbf{I}_{k-r} + \mathbf{A}_{22}\mathbf{T}_{22}|^{-\alpha} \left|\mathbf{I}_{k-r} + \mathbf{B}_{22}^2\mathbf{T}_{22}\right|^{\alpha - \frac{r}{2}},$$

where $\mathbf{B}_{22}$*,* $\mathbf{B}_{12}$ *and* $\mathbf{B}_{21}$ *are the blocks of* $\mathbf{A}^{1/2}$*.*

Remark 19. *Note that since* $r = \lceil 2\alpha \rceil$*, so that* $r - 1 < 2\alpha \le r$ *and thus* $\alpha > (r-1)/2$*, the first factor in the LT of* $\mathbf{X}_{11}$ *is the LT of a non-singular MG variable* $\mathbf{\Gamma}_1 \sim \mathcal{MG}_r(\alpha, \mathbf{A}_{11})$*. If, in addition, the matrix* $\mathbf{B}_{12}\mathbf{B}_{21}$ *is positive definite, then we have the following distributional identity*

$$\mathbf{X}_{11} + \mathbf{\Gamma}_0 \stackrel{d}{=} \mathbf{\Gamma}_1 + \boldsymbol{W},$$

where all the variables are mutually independent, $\mathbf{\Gamma}_0 \sim \mathcal{MG}_r(\alpha, \mathbf{B}_{12}\mathbf{B}_{21})$ *is non-singular MG, and* $\boldsymbol{W} \sim \mathcal{W}_r(r, \mathbf{B}_{12}\mathbf{B}_{21}/2)$ *is non-singular Wishart. This is a consequence of Theorem 3.14, which implies the identity*

$$\psi_{\mathbf{X}_{11}}(\mathbf{T}_{11}) \, |\mathbf{I}_r + \mathbf{B}_{12}\mathbf{B}_{21}\mathbf{T}_{11}|^{-\alpha} = |\mathbf{I}_r + \mathbf{A}_{11}\mathbf{T}_{11}|^{-\alpha} \, |\mathbf{I}_r + \mathbf{B}_{12}\mathbf{B}_{21}\mathbf{T}_{11}|^{-\frac{r}{2}},$$

where the four factors in the above relation are the LTs corresponding to the four relevant distributions. Further, if $\alpha < 1/2$*, so that* $r = 1$*, all these four variables have univariate gamma distributions. In particular, the distribution of the variable* $\mathbf{X}_{11}$ *is a convolution of two gamma distributions, say* $\mathcal{G}(\alpha, a)$ *and* $\mathcal{G}(1/2-\alpha, b)$*, while* $\Gamma_0 \sim \mathcal{G}(\alpha, b)$*, and thus the distribution of the left-hand-side in the distributional identity becomes the convolution of* $\mathcal{G}(\alpha, a)$ *and* $\mathcal{G}(\alpha, 1/2)$*, which is precisely the distribution of the right-hand-side. There is a similar interpretation involving the variable* $\mathbf{X}_{22}$*. However, in order for the first factor in the LT of* $\mathbf{X}_{22}$ *to be the LT of a non-singular MG variable* $\widetilde{\mathbf{\Gamma}}_1 \sim \mathcal{MG}_{k-r}(\alpha, \mathbf{A}_{22})$*, we need to have that* $\alpha > (k-r-1)/2$*. It can be shown that this condition is satisfied whenever* $\alpha > (\lceil k/2 \rceil - 1)/2$*, in which case we have the distributional identity*

$$\mathbf{X}_{22} + \widetilde{\mathbf{\Gamma}}_0 \stackrel{d}{=} \widetilde{\mathbf{\Gamma}}_1 + \widetilde{\boldsymbol{W}},$$

where all the variables are mutually independent, $\widetilde{\mathbf{\Gamma}}_0 \sim \mathcal{MG}_{k-r}(\alpha, \mathbf{B}_{22}^2)$ *is non-singular MG, and* $\widetilde{\boldsymbol{W}} \sim \mathcal{W}_{k-r}(r, \mathbf{B}_{22}^2/2)$ *is non-singular Wishart. Again, this follows from Theorem 3.14, which implies the identity*

$$\psi_{\mathbf{X}_{22}}(\mathbf{T}_{22}) \left|\mathbf{I}_{k-r} + \mathbf{B}_{22}^2\mathbf{T}_{22}\right|^{-\alpha} = |\mathbf{I}_{k-r} + \mathbf{A}_{22}\mathbf{T}_{22}|^{-\alpha} \left|\mathbf{I}_{k-r} + \mathbf{B}_{22}^2\mathbf{T}_{22}\right|^{-\frac{r}{2}},$$

where the four factors in the above relation are the LTs corresponding to the four relevant distributions. Further, if $(k-2)/2 < \alpha < (k-1)/2$*, so that* $r = k-1$ *and* $k - r = 1$*, all the variables above have univariate gamma distributions, with the distribution of* $\mathbf{X}_{22}$ *being a convolution of two gamma distributions, say* $\mathcal{G}(\alpha, a)$ *and* $\mathcal{G}(r/2 - \alpha, b)$ *and where* $\widetilde{\Gamma}_0 \sim \mathcal{G}(\alpha, b)$*. It follows that the distribution of the left-hand-side in the distributional identity becomes the convolution of* $\mathcal{G}(\alpha, a)$ *and* $\mathcal{G}(\alpha, r/2)$*, which is precisely the distribution of the right-hand-side.*

3.4. **The scaling $\mathbf{L} = \mathbf{A}_{2\cdot 1}$.** In this section, we propose an alternative singular MG distribution with a dispersion matrix parameter. While this family is essentially different from the one discussed in the previous section, the two families coincide in several important special cases.

Definition 4. *A $k \times k$ random matrix $\mathbf{X}$ is said to have a singular matrix-variate gamma distribution with parameter $\alpha \in (0, (k-1)/2]$ and dispersion matrix $\mathbf{A}$ if the following stochastic representation holds*

$$\mathbf{X} \stackrel{d}{=} \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{X}_{12}^\top & \mathbf{X}_{12}^\top \mathbf{X}_{11}^{-1} \mathbf{X}_{12} \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}. \tag{3.14}$$

Here, $\mathbf{X}_{11} \sim \mathcal{MG}_r(\alpha, \mathbf{A}_{11})$ and $\mathbf{X}_{12}^\top | \mathbf{X}_{11} \sim \mathcal{MN}_{k-r,r}\left(\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{X}_{11}, \frac{1}{2}\mathbf{A}_{22\cdot 1} \otimes \mathbf{X}_{11}\right)$, where $\dim(\mathbf{X}_{11}) = \dim(\mathbf{A}_{11}) = r \times r$ with $r = \lceil 2\alpha \rceil$.

We consider this distribution as a most natural extension with scaling of the non-singular matrix gamma distribution and thus we consistently denote it by $\mathcal{MG}_k(\alpha, \mathbf{A})$. Observe that, as in the standard case corresponding to $\mathbf{A} = \mathbf{I}_k$, the above definition can be retained for any $r \leq \lceil 2\alpha \rceil$. However, as in the standard case, we restrict ourselves to the largest r, where $r = \lceil 2\alpha \rceil$. We also note that the distributions defined above constitute a subclass of the scaled singular MG distributions. The stochastic representation discussed in the result below follows directly from Definition 4.

Theorem 3.15. *The class of distributions defined in Definition 4 coincides with the $\mathcal{SMG}_k(\alpha, \mathbf{L})$ distributions with $\mathbf{L} = \mathbf{A}_{2\cdot 1}$ given by (3.11). In addition, under the assumptions of Proposition 3.9, the following representation holds*

$$\mathbf{X} \stackrel{d}{=} \begin{pmatrix} \mathbf{A}_{11}^{1/2} & \mathbf{0} \\ \mathbf{A}_{21}\mathbf{A}_{11}^{-1/2} & \mathbf{A}_{22\cdot 1}^{1/2} \end{pmatrix} \begin{pmatrix} \mathbf{\Gamma}_0 & \frac{1}{\sqrt{2}}\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0^\top \\ \frac{1}{\sqrt{2}}\mathbf{Z}_0\mathbf{\Gamma}_0^{1/2} & \frac{1}{2}\mathbf{Z}_0\mathbf{Z}_0^\top \end{pmatrix} \begin{pmatrix} \mathbf{A}_{11}^{1/2} & \mathbf{A}_{21}\mathbf{A}_{11}^{-1/2} \\ \mathbf{0} & \mathbf{A}_{22\cdot 1}^{1/2} \end{pmatrix}.$$

Remark 20. *Considering the marginal distributions of $\mathbf{X} \sim \mathcal{SMG}_k(\alpha, \mathbf{A}^{1/2})$ provided by Theorem 3.14, it is clear from the above definition that the two distributions do not coincide as long as $\mathbf{B}_{12} \neq \mathbf{0}$ and $\alpha \notin S_k$, where $S_k = ((k-1)/2, \infty) \cup \{i/2, i = 1, \ldots, k-1\}$ (see the notation in Theorem 3.13). On the other hand, if $\alpha \in S_k$ or $\mathbf{A}_{12} = \mathbf{0}$, the two classes of distributions are the same.*

Remark 21. *The matrix-variate distribution $\mathcal{MG}_k(\alpha, \mathbf{A})$ with $\alpha \leq (k-1)/2$ can be naturally referred to as the singular MG distribution of rank $\lceil 2\alpha \rceil$. Let us note that the random matrix $\mathbf{X}$ with singular MG distribution is written in terms of its functionally independent elements, $\mathbf{X}_{11}$ and $\mathbf{X}_{12}$. Additionally, Definition 4 tells us that $\mathbf{X}_{11}$ has a non-singular MG distribution, while the conditional distribution of $\mathbf{X}_{12}^\top$ given $\mathbf{X}_{11}$ is a non-singular matrix-variate normal, which is consistent with the non-singular case reported in Proposition 2.3.*

Next, we point out that the singular Wishart distribution relates in the same manner to the singular MG distribution as it does in the non-singular case. The result below follows directly from Definition 2 and Corollary 3.4 of [41].

Corollary 3.16. *If $r = 2\alpha \leq k-1$ is an integer, then the $\mathcal{MG}_k\left(\frac{r}{2}, 2\mathbf{A}\right)$ distribution coincides with the singular Wishart distribution $\mathcal{W}_k(r, \mathbf{A})$.*

We now provide the density function of the singular MG distribution, with the proof in the Appendix.

Theorem 3.17. *Let $\mathbf{X} \sim \mathcal{MG}_k(\alpha, \mathbf{A})$, where $\alpha \leq (k-1)/2$ and $\mathbf{X}$, $\mathbf{A}$ are partitioned as in (3.14). Then the support of the PDF of $\mathbf{X}$ is the subset of non-negative definite $k \times k$ matrices $\mathbf{X}$ of rank $r = \lceil 2\alpha \rceil$ having the form*

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{X}_{12}^\top & \mathbf{X}_{12}^\top \mathbf{X}_{11}^{-1} \mathbf{X}_{12} \end{pmatrix},$$

where $\dim(\mathbf{X}_{11}) = r \times r$. *Moreover, the PDF is given by*

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{X}_{11}, \mathbf{X}_{12}) &= \frac{\pi^{r(r-k)/2}}{\Gamma_r(\alpha)|\mathbf{A}_{11}|^\alpha |\mathbf{A}_{22\cdot 1}|^{r/2}} |\mathbf{X}_{11}|^{\alpha-(k+1)/2} \mathrm{etr}\left\{-\mathbf{A}^{-1}\mathbf{X}\right\} \\ &= \frac{\pi^{r(r-k)/2}}{\Gamma_r(\alpha)|\mathbf{A}|^\alpha |\mathbf{A}_{22\cdot 1}|^{r/2-\alpha}} |\mathbf{X}_{11}|^{\alpha-(k+1)/2} \mathrm{etr}\left\{-\mathbf{A}^{-1}\mathbf{X}\right\}. \end{aligned} \tag{3.15}$$

In the next theorem, we consider the LT of the singular Wishart and MG distributions. The result relates to infinite divisibility and Eaton's conjecture (see [9]), the solution of which is given in [33] and [11]. We return to the discussion of infinite divisibility in Section 5. The result below follows directly from Proposition 3.10 upon noticing that for

$$\mathbf{L} = \begin{pmatrix} \mathbf{A}_{11}^{1/2} & \mathbf{0} \\ \mathbf{A}_{21}\mathbf{A}_{11}^{-1/2} & \mathbf{A}_{22\cdot 1}^{1/2} \end{pmatrix}$$

and a $k \times k$ matrix $\mathbf{T}$ we have

$$\left(\mathbf{L}^\top \mathbf{T} \mathbf{L}\right)_{22} = \mathbf{A}_{22\cdot 1}^{1/2} \mathbf{T}_{22} \mathbf{A}_{22\cdot 1}^{1/2},$$

followed by Sylvester's determinant identity.

Theorem 3.18. *Under the setting and assumptions of Theorem 3.11, the LT of $\mathbf{X}$, evaluated at a symmetric, positive definite $\mathbf{T}$, is given by*

$$\psi_{\mathbf{X}}(\mathbf{T}) = |\mathbf{I}_k + \mathbf{A}\mathbf{T}|^{-\alpha} |\mathbf{I}_{k-r} + \mathbf{A}_{22\cdot 1}\mathbf{T}_{22}|^{\alpha - \frac{r}{2}}. \tag{3.16}$$

Corollary 3.19. *We note the following restricted LT of $\mathbf{X}$*

$$\psi_{\mathbf{X}}(\mathbf{T}) = |\mathbf{I}_k + \mathbf{A}\mathbf{T}|^{-\alpha}, \tag{3.17}$$

where

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{12}^\top & \mathbf{T}_{22} \end{pmatrix} \tag{3.18}$$

with $\mathbf{T}_{22} = \mathbf{0}$ and a symmetric $\mathbf{T}_{11}$ of dimension $r \times r$. Moreover, since the lower-right block $\mathbf{X}_{22}$ is functionally dependent on the remaining blocks of $\mathbf{X}$, this restricted LT uniquely defines the distribution of $\mathbf{X}$.

Corollary 3.20. *Under the setting and assumption of Theorem 3.11 with $\mathbf{A}_{12} = \mathbf{A}_{21}^\top = \mathbf{0}$, the LT of $\mathbf{X}$ is given by*

$$\psi_{\mathbf{X}}(\mathbf{T}) = |\mathbf{I}_k + \mathbf{A}\mathbf{T}|^{-\alpha} |\mathbf{I}_{k-r} + \mathbf{A}_{22}\mathbf{T}_{22}|^{\alpha - r/2}.$$

4. An exchangeable extension of the singular matrix-variate gamma case

The singular MG distributions introduced in the previous section lack the exchangeability property, described below. Let $\pi \in \mathcal{P}_k$, where $\mathcal{P}_k$ stands for a set of all permutations of the set $\{1, \ldots, k\}$. We have seen in Corollary 2.2 that the non-singular gamma matrices have a natural permutation invariance. However, this property does not generally hold in the *singular* MG case. To see this, consider the standard 2×2 singular lower-right and upper-left MG variables, which can be represented as

$$\begin{pmatrix} \Gamma & \sqrt{\Gamma/2}Z \\ \sqrt{\Gamma/2}Z & Z^2/2 \end{pmatrix} \text{ and } \begin{pmatrix} Z^2/2 & \sqrt{\Gamma/2}Z \\ \sqrt{\Gamma/2}Z & \Gamma \end{pmatrix}, \tag{4.1}$$

respectively, where Γ has a standard gamma distribution with shape parameter $\alpha \le 1/2$ and Z is standard normal, independent of Γ (see Proposition 3.2). While the two random matrices in (4.1) can be obtained by permutations of one another in the above sense, their distributions are clearly different for any α strictly less than $1/2$.

As seen in this example, the two-dimensional case contains two non-equivalent sub-cases of singular MG distributions for $\alpha < 1/2$. The situation is even more complex in higher dimensions. For example, in three dimensions, we have three different classes of singular MG distributions when $\alpha \in (1/2, 1)$ and $3!$ different classes when $\alpha \in (0, 1/2)$. We claim that in general, when the dimension is equal to k, there are $k!/i!$ different singular MG distributions when $\alpha \in \left(\frac{i}{2}, \frac{i+1}{2}\right)$ for $i = 0, \dots, k-1$. It is quite remarkable that for $\alpha = i/2$, $i = 1, \dots, k-1$, all these different classes collapse to one (singular) Wishart distribution, which is, in fact, exchangeable in the above sense. In the remainder of this section, we explore various modifications of the definition of the singular MG distribution that retain the exchangeability property. For simplicity, we reduce our considerations to the standard singular gamma case, where the dispersion parameter is an identity matrix.

The two-dimensional case offers an elegant way to obtain the exchangeability through Corollary 2.7. The basic properties of this method are summarized in the following result, which follows easily from the properties of the singular MG distributions given in Proposition 3.2.

Proposition 4.1. *For $\alpha \in [0, 1/2]$, define a 2×2 matrix-variate distribution through the following representation*

$$\mathbf{X} = \mathbf{X}_0 + \begin{pmatrix} \Gamma_{1/2-\alpha} & 0 \\ 0 & 0 \end{pmatrix},$$

where the $\mathbf{X}_0 \sim \mathcal{MG}_2(\alpha)$ is independent of the standard gamma distributed variable $\Gamma_{1/2-\alpha}$ with shape parameter $1/2 - \alpha$. Then the following properties hold

(i) *The distribution of $\mathbf{X}$ is exchangeable, that is*

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} X_{22} & X_{21} \\ X_{12} & X_{11} \end{pmatrix}.$$

(ii) *The LT of $\mathbf{X}$, evaluated at a symmetric matrix $\mathbf{T} = (t_{ij})_{i,j=1}$, is given by*

$$\psi(\mathbf{T}) = \left((1+t_{11})(1+t_{22}) - t_{12}^2\right)^{-\alpha} \left((1+t_{11})(1+t_{22})\right)^{\alpha-1/2}.$$

(iii) *The following representations hold*

$$\mathbf{X} \stackrel{d}{=} \begin{pmatrix} Z_{1/2} & 0 \\ Z_\alpha & Z_{1/2-\alpha} \end{pmatrix} \begin{pmatrix} Z_{1/2} & Z_\alpha \\ 0 & Z_{1/2-\alpha} \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} Z_{1/2-\alpha} & Z_\alpha \\ 0 & Z_{1/2} \end{pmatrix} \begin{pmatrix} Z_{1/2-\alpha} & 0 \\ Z_\alpha & Z_{1/2} \end{pmatrix}.$$

(iv) *We have the following inverse relations*

$$\left(\mathbf{X}_0, \Gamma_{\frac{1}{2}-\alpha}\right) \stackrel{d}{=} \left(\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & \frac{X_{21}X_{12}}{X_{11}} \end{pmatrix}, X_{22} - \frac{X_{21}X_{12}}{X_{11}}\right)$$
$$\stackrel{d}{=} \left(\begin{pmatrix} X_{22} & X_{21} \\ X_{12} & \frac{X_{12}X_{21}}{X_{22}} \end{pmatrix}, X_{11} - \frac{X_{21}X_{12}}{X_{22}}\right).$$

Unfortunately, this approach does not generalize to an arbitrary dimension. Instead, to obtain exchangeability one can follow a generic alternative approach based on mixing through random permutations of the coordinates. Namely, let $\mathbf{\Pi}$ be a random matrix transformation corresponding to a random permutation, which is uniformly distributed over the set $\mathcal{P}_k$ of permutations, and let $\mathbf{X} \sim \mathcal{MG}_k(\alpha)$ with $\alpha < \frac{k-1}{2}$. Then, an exchangeable version of $\mathcal{MG}_k(\alpha)$ is obtained as the distribution of

$$\tilde{\mathbf{X}} = \mathbf{\Pi}\mathbf{X}.$$

The properties of $\tilde{\mathbf{X}}$ that follow directly from this definition are straightforward, and thus we omit them here. We only point out that this distribution is no longer singular, and, in the two-dimensional case, does not coincide with the one discussed in Proposition 4.1.

5. Infinite divisibility and convolutions properties

The results on the LTs of MG distributions extend the result given in [6], who considered the case of singular Wishart distribution. In particular, it follows from Corollary 1 of [6] that the LT of the k-dimensional singular Wishart matrix $\mathbf{X}$ with n degrees of freedom with $n < k$ (which corresponds to $\alpha = n/2 \leq (k-1)/2$) and covariance matrix $\mathbf{A} > 0$, i.e. $\mathbf{X} \sim \mathcal{W}_k(n, \mathbf{A})$, is given by

$$\psi_{\mathbf{X}}(\mathbf{T}) = |\mathbf{I}_k + \mathbf{A}\mathbf{T}|^{-\alpha}, \tag{5.1}$$

where $\mathbf{T} = (t_{ij})_{i,j=1,\dots,k}$ with $t_{ij} = 0$ for $i, j = n+1, \dots, k$. However, as can be seen from Proposition 3.10 and Theorems 3.13 and 3.18, the requirement that $t_{ij} = 0$ for $i, j = n+1, \dots, k$ is not really necessary. In fact, the ChF corresponding to the LT in (5.1) with an unrestricted $\mathbf{T}$ was investigated in the past. In particular, it was shown in [33] that this function corresponds to a probability distribution if and only if it is either singular Wishart or non-singular gamma. Indeed, our results in Section 3 show that for a non-singular MG (but not Wishart), assuming $t_{ij} = 0$ for $i, j = n+1, \dots, k$ in the LT does not uniquely define a distribution even if it is singular (see Theorems 3.13 and 3.18).

These observations lead directly to the following semi-group property for general MG distributions that originally was shown in [13] through a more analytical argument, see also [23]. In the result below, the only case not covered in the previous discussion is one with $\alpha = i/2$, $i = 1, \dots, k-1$, and $\beta \geq (k-1)/2$. However, the latter follows from the LT (5.1) of singular Wishart distribution.

Corollary 5.1. *The semi-group relation*

$$\mathcal{MG}_k(\alpha, \mathbf{A}) * \mathcal{MG}_k(\beta, \mathbf{A}) \stackrel{d}{=} \mathcal{MG}_k(\alpha + \beta, \mathbf{A})$$

holds for any $\alpha, \beta \in \{i/2, i = 1, \dots, k-1\} \cup ((k-1)/2, \infty)$.

We conclude that the class of (non-singular) MG distributions with a fixed dispersion matrix parameter is closed under convolutions and that the convolution property extends beyond the non-singular case to include the singular Wishart case as well. Nevertheless, the MG distribution is not infinitely divisible, which is well known, discussed in the introduction from a historical perspective.

In another line of work, the problem was also studied for the vector variate gamma distribution, and the sufficient and necessary condition for infinite divisibility in terms of the matrix scale parameter $\mathbf{A}$ has been best described in [2]. Since the class of multivariate gamma distributions is specified in a simple manner by the LT written in the form (5.1), irrespectively if one deals with matrices or vectors ($\mathbf{T}$ must be diagonal in the latter case), the conditions for infinite divisibility can be easily confused. Thus let us emphasize that the infinite divisibility in the *vector gamma case* holds if and only if the matrix $\mathbf{A}$ satisfies the following condition:

There is a diagonal matrix $\mathbf{D}$ with diagonal elements 1 or -1, such that $(\mathbf{DAD})^{-1}$ has non-positive off-diagonal elements.

Since this condition could be formulated without any changes for the matrix-variate case as well, it was mistakenly regarded as the solution to the problem of infinite divisibility, see, for example, [8]. In the following remark, we present an argument showing that such a direct matrix-variate extension of Bapat's theorem does not hold, as expected.

Remark 22. *Let us note that the identity matrix $\mathbf{D} = \mathbf{I}_k$ satisfies the above Bapat's condition. Consequently, for each $\alpha > 0$, the function $\mathbf{t} \to |\mathbf{I}_k + diag(\mathbf{t})|^{-\alpha}$ of a vector argument $\mathbf{t} \in \mathbb{R}_+^k$ is a genuine LT of a vector variate probability distribution (describing a random vector of k IID standard gamma variables with shape parameter α). However, if this was also the case for a matrix-variate argument $\mathbf{T} \in \mathbb{S}_k^+$, then Proposition 2.1 (with $\mathbf{A} = \mathbf{I}_k$ and $\mathbf{L} = \mathbf{A}^{1/2}$) would imply that the function $\mathbf{T} \to |\mathbf{I}_k + \mathbf{TA}|^{-\alpha}$ corresponded to a probability distribution on the cone $\mathbb{S}_k^+$ for any $\alpha > 0$ and $\mathbf{A} \in \mathbb{S}_k^+$. This, however, would be contradictory to the above Bapat's condition if the latter was true for the matrix case.*

We show below that the lack of infinite divisibility can be mitigated through a certain modification of the convolution property, which is quite natural and follows from the derived form of the LT of the MG family. This modification leads to a semi-group of distributions on the set of positive definite matrices with respect to the shape parameter α, which mimics the classical infinitely divisible set-up.

We first note that Corollary 2.5 holds for each $\alpha > 0$ if $\mathbf{A}_{12} = \mathbf{A}_{21}^\top = \mathbf{0}$. In order not to be distracted by matrix scaling parameters, we formulate the results below for the standard MG distribution, where $\mathbf{A} = \mathbf{I}_k$. Further, we let $\mathcal{MG}_k^r(\alpha)$ denote the distribution of a $k \times k$ random matrix that has $\mathcal{MG}_{k-r}(\alpha)$ distribution in the lower-right $(k-r) \times (k-r)$ block and zeros everywhere else. Note that for $k > 1$, we have the following distributional behavior as α approaches zero

$$\lim_{\alpha \to 0^+} \mathcal{MG}_k(\alpha) = \mathcal{MG}_k^1(1/2),$$

i.e. the lower-right $(k-1)\times(k-1)$ corner has the Wishart distribution $\mathcal{W}_{k-1}(1, \mathbf{I}_{k-1}/2)$. Accordingly, the distribution on the right-hand-side above is considered to be a MG distribution with $\alpha = 0$, that is $\mathcal{MG}_k(0) = \mathcal{MG}_k^1(1/2)$.

Theorem 5.2. *In the above notation, the following convolution property holds for the standard $\mathcal{MG}_k(\alpha)$ ($\alpha > 0$) family*

$$\mathcal{MG}_k(\alpha) * \mathcal{MG}_k(\beta) * \mathcal{F}_k^L = \mathcal{MG}_k(\alpha+\beta) * \mathcal{F}_k^R, \tag{5.2}$$

where

$$\mathcal{F}_k^L = \mathop{\Huge *}_{i=0}^{\lfloor \frac{k-r_{\alpha+\beta}-1}{2} \rfloor} \mathcal{MG}_k^{r_{\alpha+\beta}+2i}\left(\frac{r_{\alpha+\beta}}{2} - \alpha - \beta\right)$$

$$* \mathop{\Huge *}_{i=0}^{\lfloor \frac{k-r_{\alpha}}{2} - 1 \rfloor} \mathcal{MG}_k^{r_\alpha+2i+1}\left(\alpha - \frac{r_\alpha - 1}{2}\right) * \mathop{\Huge *}_{i=0}^{\lfloor \frac{k-r_{\beta}}{2} - 1 \rfloor} \mathcal{MG}_k^{r_\beta+2i+1}\left(\beta - \frac{r_\beta - 1}{2}\right)$$

$$\mathcal{F}_k^R = \mathop{\Huge *}_{i=0}^{\lfloor \frac{k-r_{\alpha+\beta}}{2} - 1 \rfloor} \mathcal{MG}_k^{r_{\alpha+\beta}+2i+1}\left(\alpha + \beta - \frac{r_{\alpha+\beta} - 1}{2}\right)$$

$$* \mathop{\Huge *}_{i=0}^{\lfloor \frac{k-r_{\alpha}-1}{2} \rfloor} \mathcal{MG}_k^{r_\alpha+2i}\left(\frac{r_\alpha}{2} - \alpha\right) * \mathop{\Huge *}_{i=0}^{\lfloor \frac{k-r_{\beta}-1}{2} \rfloor} \mathcal{MG}_k^{r_\beta+2i}\left(\frac{r_\beta}{2} - \beta\right),$$

r_a is set to $\lceil 2a \rceil$ for $2a \le k-1$ and to $2a$ otherwise, and the $\lceil \cdot \rceil$, $\lfloor \cdot \rfloor$ are the ceiling and the floor functions, respectively. Moreover, the convolution operator over an empty set of the indices is assumed to yield a degenerated distribution sitting on the $k \times k$ matrix of zeros. We also assume that whenever dimensions of the matrices in the above convolution products are not interpretable the corresponding term represents the degenerated distribution residing on the $k \times k$ zero matrix.

Let us take a closer look at the special case $k = 2$, where we have four distinct cases:
(i) $r_\alpha = r_\beta = r_{\alpha+\beta} = 1$, leading to

$$\mathcal{F}_2^L = \mathcal{MG}_2^1\left(\frac{1}{2} - \alpha - \beta\right), \ \mathcal{F}_2^R = \mathcal{MG}_2^1\left(\frac{1}{2} - \alpha\right) * \mathcal{MG}_2^1\left(\frac{1}{2} - \beta\right);$$

(ii) $r_\alpha = r_\beta = 1$ while $r_{\alpha+\beta} > 1$, so that

$$\mathcal{F}_2^L = \delta_{\mathbf{0}}, \ \mathcal{F}_2^R = \mathcal{MG}_2^1\left(\frac{1}{2} - \alpha\right) * \mathcal{MG}_2^1\left(\frac{1}{2} - \beta\right),$$

where $\delta_{\mathbf{0}}$ is a degenerated distribution at $\mathbf{0}$;
(iii) Only one of r_α and r_β is equal to one, while the other one is greater than 1, in which case we also have $r_{\alpha+\beta} > 1$. For example, if $r_\beta = 1$, we have

$$\mathcal{F}_2^L = \delta_{\mathbf{0}}, \ \mathcal{F}_2^R = \mathcal{MG}_2^1\left(\frac{1}{2} - \beta\right);$$

(iv) All three values are greater than 1, in which case we have $\mathcal{F}_2^L = \mathcal{R}_2^L = \delta_{\mathbf{0}}$ and the regular convolution property holds.

We summarize this special case in the result below, where, without loss of generality, we assume that $\alpha \ge \beta$.

Corollary 5.3. *The case of $k=2$ with $\alpha \geq \beta$ yields the following*

(5.3)
$$\mathcal{MG}_2(\alpha)*\mathcal{MG}_2(\beta) = \mathcal{MG}_2(\alpha+\beta)*\begin{cases}\delta_{\mathbf{0}} & \text{for } \alpha,\beta \geq \frac{1}{2}\\ \mathcal{MG}_2(0) & \text{for } \alpha+\beta \leq \frac{1}{2}\\ \mathcal{MG}_2^1(1-\alpha-\beta) & \text{for } \frac{1}{4} < \alpha \leq \frac{1}{2}, \alpha+\beta > \frac{1}{2}\\ \mathcal{MG}_2^1(\frac{1}{2}-\beta) & \text{for } \beta \leq \frac{1}{2}, \alpha > \frac{1}{2}.\end{cases}$$

Remark 23. *Note that when $\alpha,\beta \geq 1/2$, we get the regular convolution property, consistent with Corollary 5.1.*

In [39], a question of infinite divisibility of the blocks in general Wishart matrices was posed. The remaining part of this section is devoted to a complete characterization of this property in terms of suitable stochastic process representation of MG distributions. This representation is most conveniently expressed in terms of an infinitely divisible matrix-variate stochastic process, which we shall introduce first.

First, we let $\mathbf{\Gamma}(\mathbf{t}) = (\Gamma_1(t_1),\dots,\Gamma_k(t_k))$, $\mathbf{t} \in \mathbb{R}^k$, where the $\{\Gamma_i(t),\, t \in \mathbb{R}\}$ are IID standard gamma motions, extended to the whole real line by setting $\Gamma_i(t) = 0$ for $t \leq 0$, $i = 1,\dots,k$. Next, we let $\{B_{i,j}(t),\, t \in \mathbb{R}_+\}$, $i,j = 1,\dots,k$, be IID standard Brownian motions. Further, we let $\mathbf{B}(\mathbf{t})$, $\mathbf{t} = (t_1,\dots,t_k) \in \mathbb{R}_+^k$, be a triangular matrix-variate Brownian motion, defined as a process having zeros on and above the main diagonal and with values of $(B_{i,j}(t_j)$ at the (i,j)th location below the diagonal, where $1 \leq j < i \leq k$. Following the standard construction in one-dimension, we now define a triangular matrix-variate Laplace motion through the subordination,

$$\mathbf{L}(\mathbf{t}) = \frac{\sqrt{2}}{2}\mathbf{B}(\mathbf{\Gamma}(\mathbf{t})),\ \ \mathbf{t} \in \mathbb{R}^k,$$

assuming that the processes $\mathbf{B}(\cdot)$ and $\mathbf{\Gamma}(\cdot)$ are independent. Finally, we define a matrix-variate stochastic process

$$\mathbf{GL}(\mathbf{t}) = \operatorname{diag}(\mathbf{\Gamma}(\mathbf{t})) + \mathbf{L}(\mathbf{t}) + \mathbf{L}(\mathbf{t})^\top,\ \ \mathbf{t} \in \mathbb{R}^k,$$

which we term a matrix *gamma-Laplace motion.* The name is justified by the following result, whose straightforward proof is given in the appendix.

Proposition 5.4. *A gamma-Laplace motion* $\mathbf{GL}$ *is a Lévy motion on $\mathbb{R}_+^k = [0,\infty)^k$, i.e. a process started at zero and with independent and homogeneous increments, satisfying the following conditions:*

(i) $\mathbf{GL}(\mathbf{0}) = \mathbf{0}$*;*
(ii) *For each m and $\mathbf{t}_i = (t_{i,1},\dots t_{i,k})^\top, \mathbf{s}_i = (s_{i,1},\dots s_{i,k})^\top \in (0,\infty)^k$, $i = 1,\dots,m$, such that $t_{i,j} \leq t_{i,j}+s_{i,j} \leq t_{i+1,j}$, $i = 1,\dots,m-1$, $j = 1,\dots,k$, the random variables $\mathbf{GL}(\mathbf{t}_i+\mathbf{s}_i) - \mathbf{GL}(\mathbf{t}_i)$ are independent, with distributions depending only on the $\{\mathbf{s}_i\}$, $i = 1,\dots,m$;*
(iii) *The LT and the ChF of $\mathbf{GL}(\mathbf{s})$, where $\mathbf{s} = (s_1,\dots,s_k)^\top \in \mathbb{R}_+^k$, evaluated at a symmetric $k\times k$ matrix $\mathbf{T} = (t_{i,j})$, $i,j = 1,\dots k$, are given by*

$$\psi_{\mathbf{GL}(\mathbf{t})}(\mathbf{T}) = \frac{1}{\prod_{l=1}^k \left(1+\tilde{\mathbf{t}}_l\tilde{\mathbf{t}}_l^\top + t_{ll}\right)^{s_l}},$$
$$\phi_{\mathbf{GL}(\mathbf{t})}(\mathbf{T}) = \frac{1}{\prod_{l=1}^k \left(1+\tilde{\mathbf{t}}_l\tilde{\mathbf{t}}_l^\top - \imath t_{ll}\right)^{s_l}},$$

respectively, where $\tilde{\mathbf{t}}_l = \left(t_{l,l+1}\, t_{l,l+2} \dots t_{l,k}\right)$ *for* $l = 1, \dots, k-1$ *and* $\tilde{\mathbf{t}}_k = 0$.

Corollary 5.5. *The distribution of* $(\mathbf{\Gamma}(\mathbf{t}), \mathbf{L}(\mathbf{t}))$ *is infinitely divisible for each* $\mathbf{t} \in \mathbb{R}^k$.

While the gamma-Laplace process is infinitely divisible, its values are not necessarily positive definite matrices. However, the following modification leads to a process having values in $\mathbb{S}_k^+$

$$\mathbf{X} = \mathbf{GL} + \operatorname{diag}(\mathbf{GL})^{-\frac{1}{2}} \operatorname{ltr}(\mathbf{GL}) \operatorname{ltr}(\mathbf{GL})^\top \operatorname{diag}(\mathbf{GL})^{-\frac{1}{2}}, \tag{5.4}$$

where $\operatorname{ltr}(\mathbf{GL})$ is a triangular matrix with the entries of $\mathbf{GL}$ below the main diagonal and zeros otherwise. The following results summarize the properties of $\mathbf{X}$ that justify using the name *matrix-variate gamma process* for $\mathbf{X}$. They can be easily verified by using standard matrix algebra.

Proposition 5.6. *Consider the matrix-variate gamma process* $\mathbf{X}$ *given by* (5.4). *Then*

$$\mathbf{X} = \operatorname{diag}(\mathbf{\Gamma}) + \left(\mathbf{L} + \mathbf{L}^\top\right) + \operatorname{diag}(\mathbf{\Gamma})^{-\frac{1}{2}} \mathbf{L}\mathbf{L}^\top \operatorname{diag}(\mathbf{\Gamma})^{-\frac{1}{2}} = \mathbf{R}\mathbf{R}^\top,$$

where $\mathbf{R}$ *is a lower triangular matrix process defined as follows*

$$\mathbf{R}(\mathbf{t}) = \operatorname{diag}(\mathbf{\Gamma})^{\frac{1}{2}} + \operatorname{diag}(\mathbf{\Gamma})^{-\frac{1}{2}} \mathbf{L}$$

$$= \begin{pmatrix} \sqrt{\Gamma_1(t_1)} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{L_{21}(t_1)}{\sqrt{\Gamma_1(t_1)}} & \sqrt{\Gamma_2(t_2)} & 0 & 0 & \dots & 0 & 0 \\ \frac{L_{31}(t_1)}{\sqrt{\Gamma_1(t_1)}} & \frac{L_{32}(t_1)}{\sqrt{\Gamma_2(t_2)}} & \sqrt{\Gamma_3(t_3)} & 0 & \dots & 0 & 0 \\ \frac{L_{41}(t_1)}{\sqrt{\Gamma_1(t_1)}} & \frac{L_{42}(t_1)}{\sqrt{\Gamma_2(t_2)}} & \frac{L_{42}(t_1)}{\sqrt{\Gamma_3(t_3)}} & \sqrt{\Gamma_4(t_4)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{L_{k-11}(t_1)}{\sqrt{\Gamma_1(t_1)}} & \frac{L_{k-12}(t_2)}{\sqrt{\Gamma_2(t_2)}} & \frac{L_{k-13}(t_3)}{\sqrt{\Gamma_3(t_2)}} & \frac{L_{k-13}(t_3)}{\sqrt{\Gamma_3(t_2)}} & \dots & \sqrt{\Gamma_{k-1}(t_{k-1})} & 0 \\ \frac{L_{k1}(t_1)}{\sqrt{\Gamma_1(t_1)}} & \frac{L_{k2}(t_2)}{\sqrt{\Gamma_2(t_2)}} & \frac{L_{k3}(t_3)}{\sqrt{\Gamma_3(t_2)}} & \frac{L_{k3}(t_3)}{\sqrt{\Gamma_3(t_2)}} & \dots & \frac{L_{kk-1}(t_{k-1})}{\sqrt{\Gamma_{k-1}(t_{k-1})}} & \sqrt{\Gamma_k(t_k)} \end{pmatrix}.$$

Due to its role in decomposing the matrix-variate gamma process, we call the process $\mathbf{R}$ *triangular matrix Rayleigh.*

Finally, the following immediate consequence of Theorem 3.5 provides a link between the process $\mathbf{X}$ defined above and MG distribution with an arbitrary shape parameter.

Corollary 5.7. *For* $t > 0$, *let* $\mathbf{t}(t) = (t, t - \frac{1}{2}, \dots, t - \frac{k-1}{2})$. *Then, we have*

$$\mathbf{X}(\mathbf{t}(t)) \sim \mathcal{MG}_k(t).$$

Appendix A. Proofs

Proof of Proposition 2.1. The result follows from the following identities

$$\begin{aligned} \psi_{\mathbf{LXL}^\top}(\mathbf{T}) &= \mathbb{E}\left[\operatorname{etr}\left\{-\mathbf{TLXL}^\top\right\}\right] = \mathbb{E}\left[\operatorname{etr}\left\{-\mathbf{L}^\top\mathbf{TLX}\right\}\right] \\ &= \left|\mathbf{I}_k + \mathbf{L}^\top\mathbf{TLA}\right|^{-\alpha} = \left|\mathbf{I}_q + \mathbf{TLAL}^\top\right|^{-\alpha}, \end{aligned}$$

where the first line follows from the properties of trace, while the second line follows from (2.2) and Sylvester's determinant theorem. □

Proof of Corollary 2.4. Let us consider the following matching of the parameters of Proposition 2.3: $v = r$, $u = k - r$, $\mathbf{A} = \mathbf{A}_{22\cdot 1}^{-1}$, $\mathbf{C} = \mathbf{x}_{21}$, and $\mathbf{B} = \mathbf{A}_{11\cdot 2}^{-1}$, where we assume that $\mathbf{X}_{12}^\top = \mathbf{x}_{12}^\top = \mathbf{x}_{21}$ is non-random. Then, by Proposition 2.3 and (2.7), $\mathbf{Y}$ is distributed the same as $\mathbf{X}_{11}|(\mathbf{X}_{12} = \mathbf{x}_{12})$, $\mathbf{Z}$ has the same distribution as $\mathbf{X}_{11\cdot 2}$, and $\mathbf{X}_{22}^{-1}|\mathbf{X}_{21} = \mathbf{x}_{21}$ has the same distribution as $\mathbf{X}$. In the latter, we use the symmetry of MG distributions with respect to permutations and apply Proposition 2.3 to the variable, where the roles of the blocks are reversed, i.e. $\mathbf{X}_{11}$ is swapped with $\mathbf{X}_{22}$ and $\mathbf{X}_{21}$ with $\mathbf{X}_{12}$. On the other hand, $\mathbf{X}_{11\cdot 2}$ is independent of $(\mathbf{X}_{22}, \mathbf{X}_{12})$ and thus

$$
\begin{aligned}
\mathbf{X}_{11}| \left(\mathbf{X}_{22} = \mathbf{x}_{22}, \mathbf{X}_{12} = \mathbf{C}^\top\right) &= \mathbf{X}_{11\cdot 2} + \mathbf{C}^\top \mathbf{x}_{22}^{-1}\mathbf{C} \\
&\stackrel{d}{=} \mathbf{Z} + \mathbf{C}^\top \mathbf{x}_{22}^{-1}\mathbf{C},
\end{aligned}
$$

where the second term on the right-hand side is non-random. Consequently,

$$
\begin{aligned}
\mathbf{Y} &\stackrel{d}{=} \mathbf{Z} + \mathbf{C}^\top \left(\mathbf{X}_{22}^{-1}|\mathbf{X}_{21} = \mathbf{C}\right)\mathbf{C} \\
&= \mathbf{Z} + \mathbf{C}^\top \left(\mathbf{X}_{22}|\mathbf{X}_{21} = \mathbf{C}\right)^{-1}\mathbf{C},
\end{aligned}
$$

where $\mathbf{X}_{22}|\mathbf{X}_{21} = \mathbf{C} \sim \mathcal{MGIG}_u(\alpha - v/2, 2\mathbf{C}\mathbf{B}\mathbf{C}^\top, 2\mathbf{A})$ and is independent of $\mathbf{Z}$. Applying (2.7) leads to the final conclusion. □

Proof of Proposition 2.6. The only relation to be shown is the distributional invariance on coordinate permutations. Let us define

$$
\begin{aligned}
\tilde{Z}_{\alpha+\beta} &= \delta\sqrt{Z_\alpha^2 + Z_\beta^2}, \\
\tilde{Z}_\alpha &= |Z_{\alpha+\beta}|\, Z_\alpha/\sqrt{Z_\alpha^2 + Z_\beta^2}, \\
\tilde{Z}_\beta &= \tilde{\delta}\, |Z_{\alpha+\beta}| \sqrt{1 - Z_\alpha^2/(Z_\alpha^2 + Z_\beta^2)},
\end{aligned}
$$

where $\delta = \operatorname{sign}(Z_{\alpha+\beta})$ and $\tilde{\delta} = \operatorname{sign}(Z_\beta)$. By direct algebra, we have

$$
\begin{pmatrix} Z_{\alpha+\beta}^2 & Z_{\alpha+\beta}Z_\alpha \\ Z_{\alpha+\beta}Z_\alpha & Z_\alpha^2 + Z_\beta^2 \end{pmatrix} = \begin{pmatrix} \tilde{Z}_\alpha^2 + \tilde{Z}_\beta^2 & \tilde{Z}_{\alpha+\beta}\tilde{Z}_\alpha \\ \tilde{Z}_{\alpha+\beta}\tilde{Z}_\alpha & \tilde{Z}_{\alpha+\beta}^2 \end{pmatrix}.
$$

It is well-known that for independent gamma $\Gamma_{\alpha+\beta}$ and beta $B_{\alpha,\beta}$ random variables, the variables $\Gamma_{\alpha+\beta}B_{\alpha,\beta}$ and $\Gamma_{\alpha+\beta}(1 - B_{\alpha,\beta})$ are independent and gamma distributed, with shape parameters α and β, respectively. Note that $\tilde{Z}_{\alpha+\beta}^2$, $\tilde{Z}_\alpha^2$, $\tilde{Z}_\beta^2$ have gamma distributions with the respective shapes, and are also mutually independent. Finally, the signs of all the variables are mutually independent, producing

$$
(Z_{\alpha+\beta}, Z_\alpha, Z_\beta) \stackrel{d}{=} (\tilde{Z}_{\alpha+\beta}, \tilde{Z}_\alpha, \tilde{Z}_\beta).
$$

This concludes the proof. □

Proof of Proposition 2.9. Without loss of generality, we assume that we deal with the zero mean entries of $\mathbf{X}$. Let $\tilde{\mathbf{X}} = \operatorname{vec} \mathbf{X}$. Then $\tilde{\mathbf{X}}\tilde{\mathbf{X}}^\top$ is made of $k \times k$ blocks $\mathbf{X}_{\cdot i}\mathbf{X}_{j\cdot}$, $i, j = 1, \ldots, k$. Thus the covariance matrix $\mathbb{E}(\tilde{\mathbf{X}}\tilde{\mathbf{X}}^\top)$ is made of the blocks $\mathbb{E}(\mathbf{X}_{\cdot i}\mathbf{X}_{j\cdot})$, $i, j = 1, \ldots, k$. By the assumption of uncorrelated entries,

$$
\mathbb{E}(\mathbf{X}_{\cdot i}\mathbf{X}_{i\cdot}) = \operatorname{diag}(\boldsymbol{\Sigma}_{\cdot i})
$$

and

$$\mathbb{E}(\mathbf{X}_{\cdot i}\mathbf{X}_{j\cdot}) = \sigma_{ij}\mathbf{S}_{ij},$$

where $\mathbf{S}_{ij}$ are defined in (2.17). Then the conclusions follows from the definition of $\mathbf{K}_k$ and a straightforward rearrangement of the above identities. □

Proof of Proposition 2.11. If $\mathbf{X} \sim \mathcal{MAL}_{r,l}(\alpha; \mathbf{A}, \mathbf{M}, \mathbf{\Sigma})$, where $\alpha > \frac{r-1}{2}$, then $\mathbf{X}|\mathbf{\Gamma}$ is matrix normal $\mathcal{MN}_{r,l}(\mathbf{\Gamma M}, \mathbf{\Gamma} \otimes \mathbf{\Sigma})$. Consequently, the LT of this conditional distribution evaluated at a $r \times l$ real matrix $\mathbf{T}$ is given by

$$\psi_{\mathbf{X}|\mathbf{\Gamma}}(\mathbf{T}) = \mathbb{E}\left(\mathrm{etr}\{-\mathbf{T}^\top\mathbf{X}\}|\mathbf{\Gamma}\right) = \mathrm{etr}\left\{-\mathbf{T}^\top\mathbf{\Gamma M} + \frac{1}{2}\mathbf{T}^\top\mathbf{\Gamma T \Sigma}\right\}, \quad \mathbf{T} \in \mathbb{R}^{r\times l},$$

which is obtained by evaluating the MGF of this conditional distribution at $-\mathbf{T}$ (or by evaluating the ChF of this conditional matrix normal distribution at $\iota\mathbf{T}$). Using standard properties of the trace operator, this formula can be written as

$$\psi_{\mathbf{X}|\mathbf{\Gamma}}(\mathbf{T}) = \mathbb{E}\left(\mathrm{etr}\{-\mathbf{T}^\top\mathbf{X}\}|\mathbf{\Gamma}\right) = \mathrm{etr}\left\{-\mathbf{M}\mathbf{T}^\top\mathbf{\Gamma} + \frac{1}{2}\mathbf{T\Sigma T}^\top\mathbf{\Gamma}\right\}, \quad \mathbf{T} \in \mathbb{R}^{r\times l}.$$

Further, since $\mathrm{tr}(\mathbf{MT}^\top\mathbf{\Gamma}) = \mathrm{tr}(\mathbf{TM}^\top\mathbf{\Gamma})$, we can also write

$$\psi_{\mathbf{X}|\mathbf{\Gamma}}(\mathbf{T}) = \mathbb{E}\left(\mathrm{etr}\{-\mathbf{T}^\top\mathbf{X}\}|\mathbf{\Gamma}\right) = \mathrm{etr}\left\{-\frac{1}{2}\left[\mathbf{MT}^\top + \mathbf{TM}^\top - \mathbf{T\Sigma T}^\top\right]\mathbf{\Gamma}\right\}, \quad \mathbf{T} \in \mathbb{R}^{r\times l}.$$

Finally, since $\psi_{\mathbf{X}}(\mathbf{T}) = \mathbb{E}\left\{\mathbb{E}\left(\mathrm{etr}\{-\mathbf{T}^\top\mathbf{X}\}|\mathbf{\Gamma}\right)\right\}$, the above leads to

$$\psi_{\mathbf{X}}(\mathbf{T}) = \mathbb{E}\left(\mathrm{etr}\left\{-\frac{1}{2}\left[\mathbf{MT}^\top + \mathbf{TM}^\top - \mathbf{T\Sigma T}^\top\right]\mathbf{\Gamma}\right\}\right),$$

which we recognize as the LT of the distribution of $\mathbf{\Gamma} \sim \mathcal{MG}_r(\alpha, \mathbf{A})$ evaluated at a symmetric matrix

$$\tilde{\mathbf{T}} = \frac{1}{2}\left[\mathbf{MT}^\top + \mathbf{TM}^\top - \mathbf{T\Sigma T}^\top\right].$$

This yields the form of the Laplace transform of $\mathbf{X}$. Indeed, by the assumption (2.18), $\tilde{\mathbf{T}}$ is within the range of the Laplace transform (see (2.2)), since

$$\mathbf{I}_r + \mathbf{A}^{1/2}\tilde{\mathbf{T}}\mathbf{A}^{1/2} = \mathbf{I}_r + \frac{\mathbf{A}^{\frac{1}{2}}(\mathbf{TM}^\top + \mathbf{MT}^\top - \mathbf{T\Sigma T}^\top)\mathbf{A}^{\frac{1}{2}}}{2}.$$

The argument for the ChF is similar.

□

Proof of Proposition 2.12. Assume first that $\alpha > (r + l - 1)/2$. Then the proof is a direct consequence of Parts (i) - (iii) and (vi) of Proposition 2.3. Namely, for the blocks in this result, we again set $\mathbf{A}$ in Proposition 2.12 to coincide with $\mathbf{A}_{11}$ of Proposition 2.3, $\mathbf{M} = \mathbf{A}_{11}^{-1}\mathbf{A}_{12}$, and $\mathbf{\Sigma} = \frac{1}{2}\mathbf{A}_{22\cdot 1}$. We first note that with any choice of $\mathbf{A}$, $\mathbf{\Sigma}$, and $\mathbf{M}$ as in Proposition 2.12, the following matrix

$$\begin{pmatrix} \mathbf{A} & \mathbf{AM} \\ \mathbf{M}^\top\mathbf{A} & 2\mathbf{\Sigma} + \mathbf{M}^\top\mathbf{AM} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{AM} \\ \mathbf{M}^\top\mathbf{A} & \mathbf{M}^\top\mathbf{AM} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{\Sigma} \end{pmatrix}$$

is a positive definite matrix that can be substituted for $\mathbf{A}$ in Proposition 2.3 (the first term in the sum is the covariance matrix of $(\mathbf{X}, \mathbf{M}^\top\mathbf{X})$, where $\mathbf{X}$ has $\mathbf{A}$ as its covariance).

Identifying $\mathbf{X}_{11}$ as $\mathbf{A}^{\frac{1}{2}}\mathbf{\Gamma}\mathbf{A}^{\frac{1}{2}}$, from Parts (i) and (iii) of Proposition 2.3, we obtain in the new notation

$$\mathbf{X}_{12} = \mathbf{A}^{\frac{1}{2}}\mathbf{\Gamma}\mathbf{A}^{\frac{1}{2}}\mathbf{M} + \left(\mathbf{A}^{\frac{1}{2}}\mathbf{\Gamma}\mathbf{A}^{\frac{1}{2}}\right)^{\frac{1}{2}}\mathbf{Z}\mathbf{\Sigma}^{\frac{1}{2}}.$$

On the other hand, noticing that $\mathbf{A}'$ coincides with $\mathbf{A}_{22}$ of Proposition 2.3 and identifying $\mathbf{X}_{22}$ there as $\mathbf{A}'^{\frac{1}{2}}\mathbf{\Gamma}'\mathbf{A}'^{\frac{1}{2}}$, $\mathbf{M}' = \mathbf{A}_{22}^{-1}\mathbf{A}_{21}$, and $\mathbf{\Sigma}' = \frac{1}{2}\mathbf{A}_{11\cdot 2}$, Parts (vi) and (ii) yield

$$\mathbf{X}_{12} = \mathbf{M}'^{\top}\mathbf{A}'^{\frac{1}{2}}\mathbf{\Gamma}'\mathbf{A}'^{\frac{1}{2}} + \mathbf{\Sigma}'^{\frac{1}{2}}\mathbf{Z}\left(\mathbf{A}'^{\frac{1}{2}}\mathbf{\Gamma}'\mathbf{A}'^{1/2}\right)^{\frac{1}{2}}$$

and thus proving the result.

To argue the result for $\alpha \in ((\min(r,l)-1)/2, (r+l-1)/2]$, we notice that from Proposition 2.11, the ChFs taken to the power $-1/\alpha$ has the same form, independently of the value of α. Thus, if the equality holds for some α it must hold for all α, which concludes the proof. □

Proof of Proposition 2.14. To establish the representation as a difference of two independent gamma matrices, we first note the following LT of $\mathbf{X} + \mathbf{X}^{\top}$, obtained from the LT of $\mathbf{\Gamma} \sim \mathcal{MG}_{2r}(\alpha)$:

$$\begin{aligned}\psi_{\mathbf{X}+\mathbf{X}^{\top}}(\mathbf{T}) &= \psi_{\mathbf{X}}(\mathbf{T}+\mathbf{T}^{\top}) = \psi_{\mathbf{\Gamma}}\left(\begin{pmatrix} \mathbf{0} & \frac{\mathbf{T}+\mathbf{T}^{\top}}{2} \\ \frac{\mathbf{T}+\mathbf{T}^{\top}}{2} & \mathbf{0}\end{pmatrix}\right) \\ &= \left|\mathbf{I}_r - \frac{\left(\mathbf{T}+\mathbf{T}^{\top}\right)^2}{2}\right|^{-\alpha}.\end{aligned}$$

On the other hand, by using the LT (2.2) of MG distribution, coupled with (2.4), we obtain the following expression for the LT of the difference of two independent MG variables evaluated at an arbitrary $r \times r$ matrix $\mathbf{T}$

$$\begin{aligned}\psi_{\mathbf{\Gamma}_r-\mathbf{\Gamma}_l}(\mathbf{T}) &= \left|\mathbf{I}_r + \frac{\mathbf{T}+\mathbf{T}^{\top}}{\sqrt{2}}\right|^{-\alpha}\left|\mathbf{I}_r - \frac{\mathbf{T}+\mathbf{T}^{\top}}{\sqrt{2}}\right|^{-\alpha} \\ &= \left|\mathbf{I}_r - \frac{\left(\mathbf{T}+\mathbf{T}^{\top}\right)^2}{2}\right|^{-\alpha}.\end{aligned}$$

□

Proof of Theorem 2.15. Let $\mathbf{Z}$ be a $k \times n$ random matrix of IID standard normal random variables, so that $\mathbf{Z} \sim \mathcal{MN}_{k,n}(\mathbf{0}, \mathbf{I}_k \otimes \mathbf{I}_n)$. Then $\mathbf{X} \overset{d}{=} \mathbf{\Sigma}^{1/2}\mathbf{Z}\mathbf{Z}^{\top}\mathbf{\Sigma}^{1/2}$. Using subindex $j\cdot$ for a $1 \times k$ matrix made of the jth row of a matrix, we have

$$\begin{aligned}X_{\pi(i)\pi(j)} &= \left(\mathbf{\Sigma}^{1/2}\mathbf{Z}\right)_{\pi(i)\cdot}\left(\left(\mathbf{\Sigma}^{1/2}\mathbf{Z}\right)_{\pi(j)\cdot}\right)^{\top} \\ &= \left(\mathbf{\Sigma}^{1/2}\right)_{\pi(i)\cdot}\mathbf{Z}\mathbf{Z}^{\top}\left(\left(\mathbf{\Sigma}^{1/2}\right)_{\pi(j)\cdot}\right)^{\top} \\ &\overset{d}{=} \left(\left(\mathbf{\Sigma}^{1/2}\right)_{\pi}\right)_{i\cdot}\mathbf{Z}\mathbf{Z}^{\top}\left(\left(\left(\mathbf{\Sigma}^{1/2}\right)_{\pi}\right)_{j\cdot}\right)^{\top},\end{aligned}$$

where the last equality stands for the equality of distributions resulting in permuting the rows of $\mathbf{Z}$. Consequently, $\mathbf{X}_\pi \sim \mathcal{W}_k(n, \mathbf{\Sigma}_1)$, where

$$\mathbf{\Sigma}_1 = \left(\mathbf{\Sigma}^{1/2}\right)_\pi \left(\left(\mathbf{\Sigma}^{1/2}\right)_\pi\right)^\top = \left(\mathbf{\Sigma}^{1/2}\right)_\pi \left(\mathbf{\Sigma}^{1/2}\right)_\pi = \mathbf{\Sigma}_\pi.$$

This concludes the proof. □

Proof of Proposition 3.2. The first part of the result follows almost immediately from Definition 2 upon noticing that

$$\mathbf{Z}\mathbf{Z}^\top = \begin{pmatrix} \mathbf{\Gamma}_0 & \frac{1}{\sqrt{2}}\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0^\top \\ \frac{1}{\sqrt{2}}\mathbf{Z}_0\mathbf{\Gamma}_0^{1/2} & \frac{1}{2}\mathbf{Z}_0\mathbf{Z}_0^\top \end{pmatrix}.$$

Next, since the rank of $\mathbf{\Gamma}_0 \sim \mathcal{MG}_r(\alpha)$ is r, so is the rank of $\mathbf{X}$. The form of the density is obtained as follows

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{X}_{11}, \mathbf{X}_{12}) &= f_{\mathbf{X}_{11}}(\mathbf{X}_{11}) f_{\mathbf{X}_{12}^\top|\mathbf{X}_{11}}(\mathbf{X}_{12}^\top|\mathbf{X}_{11}) \\ &= \frac{|\mathbf{X}_{11}|^{\alpha-(r+1)/2}}{\Gamma_r(\alpha)} \mathrm{etr}\{-\mathbf{X}_{11}\} \frac{2^{r(k-r)/2}}{(2\pi)^{r(k-r)/2}} |\mathbf{X}_{11}|^{-(k-r)/2} \,\mathrm{etr}\left\{-\mathbf{X}_{11}^{-1}\mathbf{X}_{12}\mathbf{X}_{21}\right\} \\ &= \frac{|\mathbf{X}_{11}|^{\alpha-(k+1)/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)} \mathrm{etr}\{-\mathbf{X}_{11}\} \mathrm{etr}\left\{-\mathbf{X}_{21}\mathbf{X}_{11}^{-1}\mathbf{X}_{12}\right\} \\ &= \frac{|\mathbf{X}_{11}|^{\alpha-(k+1)/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)} \mathrm{etr}\left\{-\mathbf{X}_{11}\right\} \mathrm{etr}\left\{-\mathbf{X}_{22}\right\}. \end{aligned}$$

We now move to the LT. Notice that

$$\begin{aligned} \psi_{\mathbf{X}}(\mathbf{T}) &= \mathbb{E}\left[\mathrm{etr}\{-\mathbf{T}\mathbf{X}\}\right] \\ &= \int_{\mathbb{S}_r^+} \frac{|\mathbf{X}_{11}|^{\alpha-(k+1)/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)} \int_{\mathcal{M}_{r,k-r}} \mathrm{etr}\left\{-(\mathbf{I}_k + \mathbf{T})\mathbf{X}\right\} \mathbf{dX}_{12}\ \mathbf{dX}_{11}, \end{aligned}$$

where $\mathcal{M}_{r,k-r}$ stands for the set of all $r \times (k-r)$ matrices. Let $\mathbf{G} = \mathbf{I}_k + \mathbf{T}$ and write

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix},$$

where $\mathbf{G}_{11}$ is an $r \times r$ matrix. Standard calculations involving properties of the trace operator and completion of the square of the difference produce

(A.1)

$$\begin{aligned} \mathrm{tr}(\mathbf{G}\mathbf{X}) &= \mathrm{tr}(\mathbf{G}_{11}\mathbf{X}_{11}) + \mathrm{tr}(\mathbf{G}_{12}\mathbf{X}_{21}) + \mathrm{tr}(\mathbf{G}_{21}\mathbf{X}_{12}) + \mathrm{tr}(\mathbf{G}_{22}\mathbf{X}_{22}) \\ &= \mathrm{tr}(\mathbf{G}_{11}\mathbf{X}_{11}) + 2\mathrm{tr}(\mathbf{G}_{12}\mathbf{X}_{21}) + \mathrm{tr}(\mathbf{G}_{22}\mathbf{X}_{21}\mathbf{X}_{11}^{-1}\mathbf{X}_{12}) \\ &= \mathrm{tr}(\mathbf{G}_{11}\mathbf{X}_{11}) + 2\mathrm{tr}(\mathbf{X}_{11}^{1/2}\mathbf{G}_{12}\mathbf{G}_{22}^{-1/2}\mathbf{G}_{22}^{1/2}\mathbf{X}_{21}\mathbf{X}_{11}^{-1/2}) + \mathrm{tr}(\mathbf{G}_{22}^{1/2}\mathbf{X}_{21}\mathbf{X}_{11}^{-1}\mathbf{X}_{12}\mathbf{G}_{22}^{1/2}) \\ &= \mathrm{tr}\left(\left(\mathbf{G}_{11} - \mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\right)\mathbf{X}_{11}\right) \\ &+ \mathrm{tr}\left(\left(\mathbf{G}_{22}^{-1/2}\mathbf{G}_{21}\mathbf{X}_{11}^{1/2} - \mathbf{G}_{22}^{1/2}\mathbf{X}_{21}\mathbf{X}_{11}^{-1/2}\right)\left(\mathbf{X}_{11}^{1/2}\mathbf{G}_{12}\mathbf{G}_{22}^{-1/2} - \mathbf{X}_{11}^{-1/2}\mathbf{X}_{12}\mathbf{G}_{22}^{1/2}\right)\right) \\ &= \mathrm{tr}\left(\mathbf{G}_{11\cdot 2}\mathbf{X}_{11}\right) + \mathrm{tr}\left(\mathbf{G}_{22}\left(\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{X}_{11} - \mathbf{X}_{21}\right)\mathbf{X}_{11}^{-1}\left(\mathbf{X}_{11}\mathbf{G}_{12}\mathbf{G}_{22}^{-1} - \mathbf{X}_{12}\right)\right). \end{aligned}$$

Consequently, we have

$$\psi_{\mathbf{X}}(\mathbf{T}) = \int_{\mathbb{S}_r^+} \frac{|\mathbf{X}_{11}|^{\alpha-(k+1)/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)} \operatorname{etr}\{-\mathbf{G}_{11\cdot 2}\mathbf{X}_{11}\}$$
$$\times \int_{\mathcal{M}_{r,k-r}} \operatorname{etr}\left\{-\mathbf{G}_{22}\left(\mathbf{X}_{21} - \mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{X}_{11}\right)\mathbf{X}_{11}^{-1}\left(\mathbf{X}_{12} - \mathbf{X}_{11}\mathbf{G}_{12}\mathbf{G}_{22}^{-1}\right)\right\} \mathbf{dX}_{12}\,\mathbf{dX}_{11}.$$

Note that the integrand in the inner integral above can be written as $Cg(\mathbf{X}_{2,1})$, where $g(\cdot)$ is the density of the (non-singular) matrix-variate normal distribution

$$\mathcal{MN}_{k-r,r}\left(\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{X}_{11}, \mathbf{G}_{22}^{-1} \otimes \mathbf{X}_{11}/2\right)$$

and

$$C = (2\pi)^{(k-r)r/2} 2^{-(k-r)r/2} |\mathbf{G}_{22}|^{-r/2} |\mathbf{X}_{11}|^{(k-r)/2}.$$

Thus, we have

$$\psi_{\mathbf{X}}(\mathbf{T}) = \pi^{(k-r)r/2}|\mathbf{G}_{22}|^{-r/2} \int_{\mathbb{S}_r^+} \frac{|\mathbf{X}_{11}|^{\alpha-(r+1)/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)} \operatorname{etr}\{-\mathbf{G}_{11\cdot 2}\mathbf{X}_{11}\}\,\mathbf{dX}_{11}$$
$$= |\mathbf{G}_{22}|^{-r/2}|\mathbf{G}_{11\cdot 2}|^{-\alpha} \int_{\mathbb{S}_r^+} \frac{|\mathbf{X}_{11}|^{\alpha-(r+1)/2}}{\Gamma_r(\alpha)|\mathbf{G}_{11\cdot 2}|^{-\alpha}} \operatorname{etr}\{-\mathbf{G}_{11\cdot 2}\mathbf{X}_{11}\}\,\mathbf{dX}_{11} = |\mathbf{G}_{22}|^{-r/2}|\mathbf{G}_{11\cdot 2}|^{-\alpha},$$

where the last step follows from the form of (non-singular) MG density, which follows from the assumption that $\mathbf{G}_{11\cdot 2} \in \mathbb{S}_r^+$. Finally, in view of $|\mathbf{I}_k + \mathbf{T}| = |\mathbf{G}_{22}||\mathbf{G}_{11\cdot 2}|$ and $\mathbf{G}_{22} = \mathbf{I}_{k-r} + \mathbf{T}_{22}$, we obtain the conclusion. The formula for the ChF can be shown along the same lines. This completes the proof. □

Proof of Proposition 3.9. The first part of the result is obvious from the definition. Next, since the rank of $\mathbf{\Gamma}_0$ is $r = \lceil 2\alpha \rceil$ and $\mathbf{L}$ is of the full rank, the rank of $\mathbf{X}$ is inherited from that of $\mathbf{U}$. For the lower-right corner of $\mathbf{X}$ to be $\mathbf{X}_{12}^\top \mathbf{X}_{11}^{-1}\mathbf{X}_{12}$, one can argue based on the following, rather obvious algebraic fact: For a matrix $\mathbf{Y}$ to have the form

$$\begin{pmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} \\ \mathbf{Y}_{12}^\top & \mathbf{Y}_{12}^\top \mathbf{Y}_{11}^{-1}\mathbf{Y}_{12} \end{pmatrix}$$

with $\mathbf{Y}_{11} \in \mathbb{S}_r^+$ it is necessary and sufficient that $\mathbf{Y} = \mathbf{Y}_0\mathbf{Y}_0^\top$, where the $k \times r$ matrix $\mathbf{Y}_0$ is given by

$$\mathbf{Y}_0 = \begin{pmatrix} \mathbf{Y}_{11}^{1/2} \\ \mathbf{Y}_{12}^\top \mathbf{Y}_{11}^{-1/2} \end{pmatrix}.$$

Thus, the second part of the result is obtained from the above with $\mathbf{Y}_0 = \mathbf{LU}$. □

Proof of Proposition 3.10. The result is obtained upon noting that $\psi_{\mathbf{X}}(\mathbf{T}) = \psi_{\tilde{\mathbf{X}}}(\mathbf{L}^\top\mathbf{TL})$, where $\tilde{\mathbf{X}} \sim \mathcal{SMG}_k(\alpha)$, utilizing the form of the LT in the standard case given in Proposition 3.2, and applying Sylvester's determinant theorem to obtain

$$|\mathbf{I}_k + \mathbf{L}^\top\mathbf{TL}|^{-\alpha} = |\mathbf{I}_k + \mathbf{LL}^\top\mathbf{T}|^{-\alpha}.$$

□

Proof of Theorem 3.11. That the distribution must have support on the set of positive definite matrices that have the specified form is clear from Proposition 3.9. For the PDF, consider first the standard case, for which we have

$$\begin{aligned}
f_{\mathbf{X}}(\mathbf{X}_{11},\mathbf{X}_{12}) &= f_{\mathbf{X}_{11}}(\mathbf{X}_{11}) f_{\mathbf{X}_{12}^\top|\mathbf{X}_{11}}(\mathbf{X}_{12}^\top|\mathbf{X}_{11})\\
&= \frac{|\mathbf{X}_{11}|^{\alpha-(r+1)/2}}{\Gamma_r(\alpha)}\mathrm{etr}\{-\mathbf{X}_{11}\}\frac{2^{r(k-r)/2}}{(2\pi)^{r(k-r)/2}}|\mathbf{X}_{11}|^{-(k-r)/2}\,\mathrm{etr}\left\{-\mathbf{X}_{11}^{-1}\mathbf{X}_{12}\mathbf{X}_{21}\right\}\\
&= \frac{|\mathbf{X}_{11}|^{\alpha-(k+1)/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)}\mathrm{etr}\{-\mathbf{X}_{11}\}\mathrm{etr}\left\{-\mathbf{X}_{21}\mathbf{X}_{11}^{-1}\mathbf{X}_{12}\right\}\\
&= \frac{|\mathbf{X}_{11}|^{\alpha-(k+1)/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)}\mathrm{etr}\left\{-\mathbf{X}_{11}\right\}\mathrm{etr}\left\{-\mathbf{X}_{22}\right\}.
\end{aligned}$$

The result holds, since in the standard case we have $\mathbf{A}=\mathbf{I}_k$ so that $|\mathbf{A}|=1$, $\mathbf{B}_{11}=\mathbf{I}_r$, and $\mathbf{B}_{12}=\mathbf{0}$.

Next, we note that there is one-to-one relation between the $(\mathbf{\Gamma}_0,\mathbf{Z}_0)$ from Proposition 3.9 and the $(\mathbf{X}_{11},\mathbf{X}_{12})$ (connected with the standard case)

$$\begin{aligned}
\mathbf{\Phi}(\mathbf{\Gamma}_0,\mathbf{Z}_0) &= (\mathbf{\Gamma}_0,\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0^\top/\sqrt{2}) = (\mathbf{X}_{11},\mathbf{X}_{12}),\\
\mathbf{\Phi}^{-1}(\mathbf{X}_{11},\mathbf{X}_{12}) &= (\mathbf{X}_{11},\sqrt{2}\mathbf{X}_{12}^\top\mathbf{X}_{11}^{-1/2}) = (\mathbf{\Gamma}_0,\mathbf{Z}_0).
\end{aligned}$$

The Jacobian $J_{\mathbf{\Phi}}$ can be obtained from the change of variable relation for the corresponding densities. Due to the independence of $\mathbf{\Gamma}_0$ and $\mathbf{Z}_0$, we have

$$\begin{aligned}
J_{\mathbf{\Phi}}(\mathbf{\Gamma}_0,\mathbf{Z}_0) &= \frac{f_{(\mathbf{\Gamma}_0,\mathbf{Z}_0)}(\mathbf{\Gamma}_0,\mathbf{Z}_0)}{f_{\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0/\sqrt{2}|\mathbf{\Gamma}_0}(\mathbf{\Phi}(\mathbf{\Gamma}_0,\mathbf{Z}_0))}\\
&= \frac{f_{\mathbf{\Gamma}_0}(\mathbf{\Gamma}_0)f_{\mathbf{Z}_0}(\mathbf{Z}_0)}{f_{\mathbf{X}_{11}}(\mathbf{\Gamma}_0)f_{\mathbf{X}_{12}|\mathbf{X}_{11}=\mathbf{\Gamma}_0}(\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0^\top/\sqrt{2})}\\
&= \frac{f_{\mathbf{Z}_0}(\mathbf{Z}_0)}{f_{\mathbf{X}_{12}|\mathbf{X}_{11}=\mathbf{\Gamma}_0}(\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0^\top/\sqrt{2})}.
\end{aligned}$$

The two densities in the above expression are given by

$$\begin{aligned}
f_{\mathbf{Z}_0}(\mathbf{Z}_0) &= (2\pi)^{-r(k-r)/2}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{Z}_0\mathbf{Z}_0^\top\right\},\\
f_{\mathbf{X}_{12}|\mathbf{X}_{11}=\mathbf{\Gamma}_0}(\mathbf{X}_{12^\text{`}}) &= (2\pi)^{-r(k-r)/2}\left|\mathbf{\Gamma}_0/2\right|^{-(k-r)/2}\mathrm{etr}\left\{-\frac{1}{2}\left(\mathbf{\Gamma}_0/2\right)^{-1}\mathbf{X}_{12^\text{`}}\mathbf{X}_{12^\text{`}}^\top\right\}.
\end{aligned}$$

Consequently, the Jacobian becomes

$$J_{\mathbf{\Phi}}(\mathbf{\Gamma}_0,\mathbf{Z}_0) = \frac{f_{\mathbf{Z}_0}(\mathbf{Z}_0)}{f_{\mathbf{X}_{12}|\mathbf{X}_{11}=\mathbf{\Gamma}_0}(\mathbf{\Gamma}_0^{1/2}\mathbf{Z}_0/\sqrt{2})} = \left|\mathbf{\Gamma}_0/2\right|^{(k-r)/2}.$$

Further, for $\mathbf{A}\in\mathbb{S}_k^+$, let us consider the general case with $\mathbf{X}\mapsto\mathbf{A}^{1/2}\mathbf{X}\mathbf{A}^{1/2}$, and its inverse $\mathbf{Y}\mapsto\mathbf{A}^{-1/2}\mathbf{Y}\mathbf{A}^{-1/2}$. We are interested in the distribution of $(\mathbf{Y}_{11},\mathbf{Y}_{12})$. Let us define $(\mathbf{\Gamma}_1,\mathbf{Z}_1)=\mathbf{\Phi}^{-1}(\mathbf{Y}_{11},\mathbf{Y}_{12})$. Then we have the following relation:

$$\begin{aligned}
f_{\mathbf{Y}_{11},\mathbf{Y}_{12}}(\mathbf{Y}_{11},\mathbf{Y}_{12}) &= f_{\mathbf{\Gamma}_1,\mathbf{Z}_1}(\mathbf{\Phi}^{-1}(\mathbf{Y}_{11},\mathbf{Y}_{12}))J_{\mathbf{\Phi}^{-1}}(\mathbf{Y}_{11},\mathbf{Y}_{12}) =\\
&= f_{\mathbf{\Gamma}_1,\mathbf{Z}_1}(\mathbf{Y}_{11},\sqrt{2}\mathbf{Y}_{11}^{-1/2}\mathbf{Y}_{12})\left|\mathbf{Y}_{11}/2\right|^{(r-k)/2}.
\end{aligned}$$

To find the distribution of $(\mathbf{\Gamma}_1,\mathbf{Z}_1)$, we note that

$$(\mathbf{\Gamma}_1,\mathbf{Z}_1) = \mathbf{\Phi}_1(\mathbf{Y}_1,\mathbf{Y}_2) = (\mathbf{Y}_1^2, 2\mathbf{Y}_2),$$

where $\tilde{\mathbf{Y}} = (\mathbf{Y}_1, \mathbf{Y}_2)$ is given through the mapping

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \mathbf{A}^{1/2} \begin{pmatrix} \mathbf{\Gamma}_0^{1/2} \\ \frac{\mathbf{Z}_0}{\sqrt{2}} \end{pmatrix}.$$

Thus, the density of $(\mathbf{\Gamma}_1, \mathbf{Z}_1)$ satisfies

$$\begin{aligned} f_{(\mathbf{\Gamma}_1,\mathbf{Z}_1)}(\mathbf{\Gamma}_1, \mathbf{Z}_1) &= f_{(\mathbf{Y}_1,\mathbf{Y}_2)}\left(\mathbf{\Phi}_1^{-1}(\mathbf{\Gamma}_1, \mathbf{Z}_1)\right) J_{\mathbf{\Phi}_1^{-1}}(\mathbf{\Gamma}_1, \mathbf{Z}_1) = \\ &= f_{(\mathbf{Y}_1,\mathbf{Y}_2)}\left(\sqrt{\mathbf{\Gamma}_1}, \mathbf{Z}_1/2\right) J_{\mathbf{\Phi}_1^{-1}}(\mathbf{\Gamma}_1, \mathbf{Z}_1) \\ &= f_{(\mathbf{Y}_1,\mathbf{Y}_2)}\left(\sqrt{\mathbf{\Gamma}_1}, \mathbf{Z}_1/2\right) J_{\mathbf{\Gamma}^{1/2}}(\mathbf{\Gamma}_1)/2^{(k-r)^2}, \end{aligned}$$

where $J_{\mathbf{\Gamma}^{1/2}}(\mathbf{\Gamma}_1)$ is the Jacobian of the transformation of a positive definite matrix $\mathbf{\Gamma}_1$ to $\mathbf{\Gamma}_1^{1/2}$. Note that

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \mathbf{A}^{1/2} \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}, \tag{A.2}$$

where $\tilde{\mathbf{X}} = (\mathbf{X}_1, \mathbf{X}_2) = \mathbf{\Phi}_1^{-1}(\mathbf{\Gamma}_0, \boldsymbol{Z}_0)$. The matrix entries in the linear transformation (A.2) are partially symmetric $(\mathbf{Y}_1)$ and partially unrestricted $(\mathbf{Y}_2)$. The Jacobian of this transformation is not given by a straightforward formula but it is still constant depending only on the matrix $\mathbf{A}$. We denote it by $J_{\mathbf{A}}$. The distribution of $(\mathbf{Y}_1, \mathbf{Y}_2)$ expresses as

$$f_{(\mathbf{Y}_1,\mathbf{Y}_2)}(\mathbf{Y}_1, \mathbf{Y}_2) = f_{\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}}\left(\mathbf{A}^{-1/2} \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}\right) J_{\mathbf{A}}.$$

Further,

$$\begin{aligned} f_{\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}}\left(\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}\right) &= f_{\mathbf{\Gamma}_0,\mathbf{Z}_0}\left(\mathbf{\Phi}_1(\mathbf{X}_1, \mathbf{X}_2)\right) J_{\mathbf{\Phi}_1}(\mathbf{X}_1, \mathbf{X}_2) \\ &= f_{\mathbf{\Gamma}_0,\mathbf{Z}_0}\left(\mathbf{X}_1^2, 2\mathbf{X}_2\right) J_{\mathbf{X}_1^2}(\mathbf{X}_1) 2^{(k-r)^2} \\ &= 2^{(k-r)^2} f_{\mathbf{\Gamma}_0}\left(\mathbf{X}_1^2\right) J_{\mathbf{X}_1^2}(\mathbf{X}_1) f_{\mathbf{Z}_0}(2\mathbf{X}_2) \\ &= 2^{(k-r)^2}(2\pi)^{-r(k-r)/2} \frac{|\mathbf{X}_1^2|^{\alpha-(r+1)/2}}{\Gamma_r(\alpha)} \operatorname{etr}\{-\mathbf{X}_1^2\} J_{\mathbf{X}_1^2}(\mathbf{X}_1) \operatorname{etr}\left\{-2\mathbf{X}_2\mathbf{X}_2^\top\right\} \\ &= \frac{2^{k^2-5kr/2+3r^2/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)} |\mathbf{X}_1^2|^{\alpha-(r+1)/2} J_{\mathbf{X}_1^2}(\mathbf{X}_1) \operatorname{etr}\{-\mathbf{X}_1^2\} \operatorname{etr}\{-2\mathbf{X}_2\mathbf{X}_2^\top\} \\ &= \frac{2^{k^2-5kr/2+3r^2/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)} |\mathbf{X}_1^2|^{\alpha-(r+1)/2} J_{\mathbf{X}_1^2}(\mathbf{X}_1) \operatorname{etr}\{-\mathbf{\Lambda}\tilde{\mathbf{X}}\tilde{\mathbf{X}}^\top\mathbf{\Lambda}\}, \end{aligned}$$

where

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \sqrt{2}\mathbf{I}_{k-r} \end{pmatrix}.$$

Thus, we have

$$\begin{aligned} f_{(\mathbf{Y}_1,\mathbf{Y}_2)}(\mathbf{Y}_1, \mathbf{Y}_2) = J_{\mathbf{A}} &\frac{2^{k^2-5kr/2+3r^2/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)} |\mathbf{B}_{11}\mathbf{Y}_1 + \mathbf{B}_{12}\mathbf{Y}_2|^{2\alpha-r-1} J_{\mathbf{X}_1^2}(\mathbf{B}_{11}\mathbf{Y}_1 + \mathbf{B}_{12}\mathbf{Y}_2) \\ &\operatorname{etr}\{-\mathbf{A}^{-1}\mathbf{\Lambda}\tilde{\mathbf{Y}}\tilde{\mathbf{Y}}^\top\mathbf{\Lambda}\}. \end{aligned}$$

This leads to

$$\begin{aligned}
f_{(\mathbf{\Gamma}_1,\mathbf{Z}_1)}(\mathbf{\Gamma}_1,\mathbf{Z}_1) &= f_{(\mathbf{Y}_1,\mathbf{Y}_2)}\left(\sqrt{\mathbf{\Gamma}_1},\mathbf{Z}_1/2\right) J_{\mathbf{\Gamma}^{1/2}}(\mathbf{\Gamma}_1)/2^{(k-r)^2}\\
&= J_{\mathbf{A}}\frac{2^{-r(k-r)/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)}|\mathbf{B}_{11}\sqrt{\mathbf{\Gamma}_1}+\mathbf{B}_{12}\mathbf{Z}_1/2|^{2\alpha-r-1}\\
&\times J_{\mathbf{X}_1^2}(\mathbf{B}_{11}\sqrt{\mathbf{\Gamma}_1}+\mathbf{B}_{12}\mathbf{Z}_1/2)J_{\mathbf{\Gamma}^{1/2}}(\mathbf{\Gamma}_1)\\
&\times \operatorname{etr}\left\{-\mathbf{A}^{-1}\mathbf{\Lambda}\begin{pmatrix}\mathbf{\Gamma}_1 & \sqrt{\mathbf{\Gamma}_1}\mathbf{Z}_1^\top/2\\ \mathbf{Z}_1\sqrt{\mathbf{\Gamma}_1}/2 & \mathbf{Z}_1\mathbf{Z}_1^\top/4\end{pmatrix}\mathbf{\Lambda}\right\}\\
&= J_{\mathbf{A}}\frac{2^{-r(k-r)/2}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)}|\mathbf{B}_{11}\sqrt{\mathbf{\Gamma}_1}+\mathbf{B}_{12}\mathbf{Z}_1/2|^{2\alpha-r-1}\\
&\times J_{\mathbf{X}_1^2}(\mathbf{B}_{11}\sqrt{\mathbf{\Gamma}_1}+\mathbf{B}_{12}\mathbf{Z}_1/2)J_{\mathbf{\Gamma}^{1/2}}(\mathbf{\Gamma}_1)\\
&\times \operatorname{etr}\left\{-\mathbf{A}^{-1}\begin{pmatrix}\mathbf{\Gamma}_1 & \sqrt{\mathbf{\Gamma}_1/2}\mathbf{Z}_1^\top\\ \mathbf{Z}_1\sqrt{\mathbf{\Gamma}_1/2} & \mathbf{Z}_1\mathbf{Z}_1^\top/2\end{pmatrix}\right\}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
f_{\mathbf{Y}_{11},\mathbf{Y}_{12}}(\mathbf{Y}_{11},\mathbf{Y}_{12}) &= f_{\mathbf{\Gamma}_1,\mathbf{Z}_1}(\mathbf{Y}_{11},\sqrt{2}\mathbf{Y}_{12}^\top\mathbf{Y}_{11}^{-1/2})\,|\mathbf{Y}_{11}/2|^{(r-k)/2}\\
&= \frac{J_{\mathbf{A}}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)}|\mathbf{B}_{11}\sqrt{\mathbf{Y}_{11}}+\mathbf{B}_{12}\mathbf{Y}_{12}^\top\mathbf{Y}_{11}^{-1/2}/\sqrt{2}|^{2\alpha-r-1}\\
&\times J_{\mathbf{X}_1^2}(\mathbf{B}_{11}\sqrt{\mathbf{Y}_{11}}+\mathbf{B}_{12}\mathbf{Y}_{12}^\top\mathbf{Y}_{11}^{-1/2}/\sqrt{2})J_{\mathbf{\Gamma}^{1/2}}(\mathbf{Y}_{11})\,|\mathbf{Y}_{11}|^{(r-k)/2}\\
&\times \operatorname{etr}\left\{-\mathbf{A}^{-1}\mathbf{Y}\right\}\\
&= \frac{J_{\mathbf{A}}}{\pi^{r(k-r)/2}\Gamma_r(\alpha)}|\mathbf{B}_{11}\mathbf{Y}_{11}+\mathbf{B}_{12}\mathbf{Y}_{12}^\top/\sqrt{2}|^{2\alpha-r-1}\\
&\times J_{\mathbf{X}_1^2}\left((\mathbf{B}_{11}\mathbf{Y}_{11}+\mathbf{B}_{12}\mathbf{Y}_{12}^\top/\sqrt{2})\mathbf{Y}_{11}^{-1/2}\right)J_{\mathbf{\Gamma}^{1/2}}(\mathbf{Y}_{11})\,|\mathbf{Y}_{11}|^{r-\alpha-(k-1)/2}\\
&\times \operatorname{etr}\left\{-\mathbf{A}^{-1}\mathbf{Y}\right\}.
\end{aligned}$$

To obtain the explicit form of the Jacobians present in the above formula we note that for the singular Wishart distribution we have an explicit form of the density. Namely, if $\alpha = r/2$ with $r < k$, then we have

$$\begin{aligned}
f_{\mathbf{Y}_{11},\mathbf{Y}_{12}}(\mathbf{Y}_{11},\mathbf{Y}_{12}) &= \frac{\pi^{r(r-k)/2}}{\Gamma_r(r/2)|\mathbf{A}_{11}|^{r/2}|\mathbf{A}_{22\cdot 1}|^{r/2}}|\mathbf{Y}_{11}|^{(r-k-1)/2}\operatorname{etr}\left\{-\mathbf{A}^{-1}\mathbf{Y}\right\}\\
&= \frac{\pi^{r(r-k)/2}}{\Gamma_r(r/2)|\mathbf{A}|^{r/2}}|\mathbf{Y}_{11}|^{(r-k-1)/2}\operatorname{etr}\left\{-\mathbf{A}^{-1}\mathbf{Y}\right\}\\
&= \frac{J_{\mathbf{A}}}{\pi^{r(k-r)/2}\Gamma_r(r/2)}|\mathbf{B}_{11}\mathbf{Y}_{11}+\mathbf{B}_{12}\mathbf{Y}_{12}^\top/\sqrt{2}|^{-1}\\
&\times J_{\mathbf{X}_1^2}\left((\mathbf{B}_{11}\mathbf{Y}_{11}+\mathbf{B}_{12}\mathbf{Y}_{12}^\top/\sqrt{2})\mathbf{Y}_{11}^{-1/2}\right)J_{\mathbf{\Gamma}^{1/2}}(\mathbf{Y}_{11})\,|\mathbf{Y}_{11}|^{(r-k+1)/2}\\
&\times \operatorname{etr}\left\{-\mathbf{A}^{-1}\mathbf{Y}\right\}.
\end{aligned}$$

From this, we deduce that

$$\frac{|\mathbf{Y}_{11}|^{-1}}{|\mathbf{A}_{11}|^{r/2}|\mathbf{A}_{22\cdot 1}|^{r/2}} = \frac{J_{\mathbf{A}}J_{\mathbf{X}_1^2}\left((\mathbf{B}_{11}\mathbf{Y}_{11}+\mathbf{B}_{12}\mathbf{Y}_{12}^\top/\sqrt{2})\mathbf{Y}_{11}^{-1/2}\right)J_{\mathbf{\Gamma}^{1/2}}(\mathbf{Y}_{11})}{|\mathbf{B}_{11}\mathbf{Y}_{11}+\mathbf{B}_{12}\mathbf{Y}_{12}^\top/\sqrt{2}|}$$

which concludes the proof. □

Proof of Theorem 3.14. The LTs of $\mathbf{X}_{11}$ and $\mathbf{X}_{22}$ are obtained by evaluating (3.13) at a matrix $\mathbf{T}$ that has zero everywhere except for the blocks $\mathbf{T}_{11}$ and $\mathbf{T}_{22}$, respectively. Straightforward algebra, involving Sylvester's determinant identity, produces

$$\begin{aligned}
\psi_{\mathbf{X}_{11}}(\mathbf{T}_{11}) &= |\mathbf{I}_r + \mathbf{A}_{11}\mathbf{T}_{11}|^{-\alpha} \left|\mathbf{I}_{k-r} + \left(\mathbf{A}^{1/2}\mathbf{T}\mathbf{A}^{1/2}\right)_{22}\right|^{\alpha-\frac{r}{2}} \\
&= |\mathbf{I}_r + \mathbf{A}_{11}\mathbf{T}_{11}|^{-\alpha} |\mathbf{I}_{k-r} + \mathbf{B}_{21}\mathbf{T}_{11}\mathbf{B}_{12}|^{\alpha-\frac{r}{2}} \\
&= |\mathbf{I}_r + \mathbf{A}_{11}\mathbf{T}_{11}|^{-\alpha} |\mathbf{I}_r + \mathbf{B}_{12}\mathbf{B}_{21}\mathbf{T}_{11}|^{\alpha-\frac{r}{2}}, \\
\psi_{\mathbf{X}_{22}}(\mathbf{T}_{22}) &= |\mathbf{I}_{k-r} + \mathbf{A}_{22}\mathbf{T}_{22}|^{-\alpha} \left|\mathbf{I}_{k-r} + \left(\mathbf{A}^{1/2}\mathbf{T}\mathbf{A}^{1/2}\right)_{22}\right|^{\alpha-\frac{r}{2}} \\
&= |\mathbf{I}_{k-r} + \mathbf{A}_{22}\mathbf{T}_{22}|^{-\alpha} \left|\mathbf{I}_{k-r} + \mathbf{B}_{22}^2\mathbf{T}_{22}\right|^{\alpha-\frac{r}{2}},
\end{aligned}$$

as desired. □

Proof of Theorem 3.17. The result follows from the standard conditioning argument and direct application of Definition 4

$$\begin{aligned}
f_{\mathbf{X}}(\mathbf{X}_{11}, \mathbf{X}_{12}) &= f_{\mathbf{X}_{11}}(\mathbf{X}_{11}) f_{\mathbf{X}_{12}|\mathbf{X}_{11}}(\mathbf{X}_{12}|\mathbf{X}_{11}) \\
&= \frac{1}{\Gamma_r(\alpha)|\mathbf{A}_{11}|^{\alpha}} |\mathbf{X}_{11}|^{\alpha-(r+1)/2} \mathrm{etr}\left\{-\mathbf{A}_{11}^{-1}\mathbf{X}\right\} f_{\mathbf{X}_{12}|\mathbf{X}_{11}}(\mathbf{X}_{12}|\mathbf{X}_{11}) \\
&= \frac{(2\pi)^{-(k-r)r/2}}{\Gamma_r(\alpha)|\mathbf{A}_{11}|^{\alpha}|\mathbf{A}_{22\cdot 1}|^{r/2}} |\mathbf{X}_{11}|^{\alpha-(k+1)/2} \mathrm{etr}\left\{-\mathbf{A}_{11}^{-1}\mathbf{X}\right\} \\
&\times \mathrm{etr}\left\{-\mathbf{X}_{11}^{-1}\left(\mathbf{X}_{12} - \mathbf{X}_{11}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right)\mathbf{A}_{22\cdot 1}^{-1}\left(\mathbf{X}_{12} - \mathbf{X}_{11}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right)^{\top}\right\} \\
&= \frac{(2\pi)^{-(k-r)r/2}}{\Gamma_r(\alpha)|\mathbf{A}_{11}|^{\alpha}|\mathbf{A}_{22\cdot 1}|^{r/2}} |\mathbf{X}_{11}|^{\alpha-(k+1)/2} \mathrm{etr}\left\{-\mathbf{A}^{-1}\mathbf{X}\right\},
\end{aligned}$$

where the last equality follows from (A.1). □

Proof of Theorem 5.2. In the proof, we utilize the LT (3.4) given in Proposition 3.2. Accordingly, the LT of the left-hand-side of (5.2) is given by

$$
\begin{aligned}
&|\mathbf{I}_k+\mathbf{T}|^{-\alpha-\beta}\,|\mathbf{I}_{k-r_\alpha}+\mathbf{T}_{22}^{r_\alpha}|^{\alpha-\frac{r_\alpha}{2}}\,\left|\mathbf{I}_{k-r_\beta}+\mathbf{T}_{22}^{r_\beta}\right|^{\beta-\frac{r_\beta}{2}}\\
&\times\prod_{i=0}^{\lfloor\frac{k-r_{\alpha+\beta}-1}{2}\rfloor}\left|\mathbf{I}_{k-r_{\alpha+\beta}-2i}+\mathbf{T}_{22}^{r_{\alpha+\beta}+2i}\right|^{\alpha+\beta-\frac{r_{\alpha+\beta}}{2}}\left|\mathbf{I}_{k-r_{\alpha+\beta}-2i-1}+\mathbf{T}_{22}^{r_{\alpha+\beta}+2i+1}\right|^{\frac{r_{\alpha+\beta}-1}{2}-\alpha-\beta}\\
&\quad\times\prod_{i=0}^{\lfloor\frac{k-r_\alpha}{2}-1\rfloor}\left|\mathbf{I}_{k-r_\alpha-2i-1}+\mathbf{T}_{22}^{r_\alpha+2i+1}\right|^{\frac{r_\alpha-1}{2}-\alpha}\left|\mathbf{I}_{k-r_\alpha-2i-2}+\mathbf{T}_{22}^{r_\alpha+2i+2}\right|^{\alpha-\frac{r_\alpha}{2}}\\
&\quad\times\prod_{i=0}^{\lfloor\frac{k-r_\beta}{2}-1\rfloor}\left|\mathbf{I}_{k-r_\beta-2i-1}+\mathbf{T}_{22}^{r_\beta+2i+1}\right|^{\frac{r_\beta-1}{2}-\beta}\left|\mathbf{I}_{k-r_\beta-2i-2}+\mathbf{T}_{22}^{r_\beta+2i+2}\right|^{\beta-\frac{r_\beta}{2}}\\
&\qquad\qquad=|\mathbf{I}_k+\mathbf{T}|^{-\alpha-\beta}\left|\mathbf{I}_{k-r_{\alpha+\beta}}+\mathbf{T}_{22}^{r_{\alpha+\beta}}\right|^{\alpha+\beta-\frac{r_{\alpha+\beta}}{2}}\\
&\qquad\qquad\quad\times\left|\mathbf{I}_{k-r_{\alpha+\beta}-1}+\mathbf{T}_{22}^{r_{\alpha+\beta}+1}\right|^{\frac{r_{\alpha+\beta}-1}{2}-\alpha-\beta}\\
&\times\prod_{i=1}^{\lfloor\frac{k-r_{\alpha+\beta}-1}{2}\rfloor}\left|\mathbf{I}_{k-r_{\alpha+\beta}-2i}+\mathbf{T}_{22}^{r_{\alpha+\beta}+2i}\right|^{\alpha+\beta-\frac{r_{\alpha+\beta}}{2}}\left|\mathbf{I}_{k-r_{\alpha+\beta}-2i-1}+\mathbf{T}_{22}^{r_{\alpha+\beta}+2i+1}\right|^{\frac{r_{\alpha+\beta}-1}{2}-\alpha-\beta}\\
&\times|\mathbf{I}_{k-r_\alpha}+\mathbf{T}_{22}^{r_\alpha}|^{\alpha-\frac{r_\alpha}{2}}\prod_{i=0}^{\lfloor\frac{k-r_\alpha}{2}-1\rfloor}\left|\mathbf{I}_{k-r_\alpha-2i-1}+\mathbf{T}_{22}^{r_\alpha+2i+1}\right|^{\frac{r_\alpha-1}{2}-\alpha}\left|\mathbf{I}_{k-r_\alpha-2i-2}+\mathbf{T}_{22}^{r_\alpha+2i+2}\right|^{\alpha-\frac{r_\alpha}{2}}\\
&\times\left|\mathbf{I}_{k-r_\beta}+\mathbf{T}_{22}^{r_\beta}\right|^{\beta-\frac{r_\beta}{2}}\prod_{i=0}^{\lfloor\frac{k-r_\beta}{2}-1\rfloor}\left|\mathbf{I}_{k-r_\beta-2i-1}+\mathbf{T}_{22}^{r_\beta+2i+1}\right|^{\frac{r_\beta-1}{2}-\beta}\left|\mathbf{I}_{k-r_\beta-2i-2}+\mathbf{T}_{22}^{r_\beta+2i+2}\right|^{\beta-\frac{r_\beta}{2}},
\end{aligned}
$$

where $\mathbf{T}_{22}^r$ is the lower-right $(k-r)\times(k-r)$ block of $\mathbf{T}$. It is now enough to show that the last four lines correspond to the LT of $\mathcal{F}_k^R$, which has the form

$$
\begin{aligned}
&\prod_{i=0}^{\lfloor\frac{k-r_{\alpha+\beta}}{2}-1\rfloor}\left|\mathbf{I}_{k-r_{\alpha+\beta}-2i-1}+\mathbf{T}_{22}^{r_{\alpha+\beta}+2i+1}\right|^{\frac{r_{\alpha+\beta}-1}{2}-\alpha-\beta}\left|\mathbf{I}_{k-r_{\alpha+\beta}-2i-2}+\mathbf{T}_{22}^{r_{\alpha+\beta}+2i+2}\right|^{\alpha+\beta-\frac{r_{\alpha+\beta}}{2}}\\
&\quad\times\prod_{i=0}^{\lfloor\frac{k-r_\alpha-1}{2}\rfloor}\left|\mathbf{I}_{k-r_\alpha-2i}+\mathbf{T}_{22}^{r_\alpha+2i}\right|^{\alpha-\frac{r_\alpha}{2}}\left|\mathbf{I}_{k-r_\alpha-2i-1}+\mathbf{T}_{22}^{r_\alpha+2i+1}\right|^{\frac{r_\alpha-1}{2}-\alpha}\\
&\qquad\times\prod_{i=0}^{\lfloor\frac{k-r_\beta-1}{2}\rfloor}\left|\mathbf{I}_{k-r_\beta-2i}+\mathbf{T}_{22}^{r_\beta+2i}\right|^{\beta-\frac{r_\beta}{2}}\left|\mathbf{I}_{k-r_\beta-2i-1}+\mathbf{T}_{22}^{r_\beta+2i+1}\right|^{\frac{r_\beta-1}{2}-\beta}.
\end{aligned}
$$

This can be written as

$$\left|\mathbf{I}_{k-r_{\alpha+\beta}-1}+\mathbf{T}_{22}^{r_{\alpha+\beta}+1}\right|^{\frac{r_{\alpha+\beta}-1}{2}-\alpha-\beta}\left|\mathbf{I}_{k-r_{\alpha+\beta}-2}+\mathbf{T}_{22}^{r_{\alpha+\beta}+2}\right|^{\alpha+\beta-\frac{r_{\alpha+\beta}-1}{2}}$$
$$\times\prod_{i=1}^{\lfloor\frac{k-r_{\alpha+\beta}}{2}-1\rfloor}\left|\mathbf{I}_{k-r_{\alpha+\beta}-2i-1}+\mathbf{T}_{22}^{r_{\alpha+\beta}+2i+1}\right|^{\frac{r_{\alpha+\beta}-1}{2}-\alpha-\beta}\left|\mathbf{I}_{k-r_{\alpha+\beta}-2i-2}+\mathbf{T}_{22}^{r_{\alpha+\beta}+2i+2}\right|^{\alpha+\beta-\frac{r_{\alpha+\beta}-1}{2}}$$
$$\times\left|\mathbf{I}_{k-r_\alpha}+\mathbf{T}_{22}^{r_\alpha}\right|^{\alpha-\frac{r_\alpha}{2}}\left|\mathbf{I}_{k-r_\alpha-1}+\mathbf{T}_{22}^{r_\alpha+1}\right|^{\frac{r_\alpha-1}{2}-\alpha}$$
$$\times\prod_{i=1}^{\lfloor\frac{k-r_\alpha-1}{2}\rfloor}\left|\mathbf{I}_{k-r_\alpha-2i}+\mathbf{T}_{22}^{r_\alpha+2i}\right|^{\alpha-\frac{r_\alpha}{2}}\left|\mathbf{I}_{k-r_\alpha-2i-1}+\mathbf{T}_{22}^{r_\alpha+2i+1}\right|^{\frac{r_\alpha-1}{2}-\alpha}$$
$$\times\left|\mathbf{I}_{k-r_\beta}+\mathbf{T}_{22}^{r_\beta}\right|^{\beta-\frac{r_\beta}{2}}\left|\mathbf{I}_{k-r_\beta-1}+\mathbf{T}_{22}^{r_\beta+1}\right|^{\frac{r_\beta-1}{2}-\beta}$$
$$\times\prod_{i=1}^{\lfloor\frac{k-r_\beta-1}{2}\rfloor}\left|\mathbf{I}_{k-r_\beta-2i}+\mathbf{T}_{22}^{r_\beta+2i}\right|^{\beta-\frac{r_\beta}{2}}\left|\mathbf{I}_{k-r_\beta-2i-1}+\mathbf{T}_{22}^{r_\beta+2i+1}\right|^{\frac{r_\beta-1}{2}-\beta}.$$

However, it is easy to notice that we have

$$\prod_{i=0}^{\lfloor\frac{k-r_\alpha}{2}-1\rfloor}\left|\mathbf{I}_{k-r_\alpha-2i-1}+\mathbf{T}_{22}^{r_\alpha+2i+1}\right|^{\frac{r_\alpha-1}{2}-\alpha}\left|\mathbf{I}_{k-r_\alpha-2i-2}+\mathbf{T}_{22}^{r_\alpha+2i+2}\right|^{\alpha-\frac{r_\alpha}{2}}$$
$$=\left|\mathbf{I}_{k-r_\alpha-1}+\mathbf{T}_{22}^{r_\alpha+1}\right|^{\frac{r_\alpha-1}{2}-\alpha}\times$$
$$\times\prod_{i=1}^{\lfloor\frac{k-r_\alpha-1}{2}\rfloor}\left|\mathbf{I}_{k-r_\alpha-2i}+\mathbf{T}_{22}^{r_\alpha+2i}\right|^{\alpha-\frac{r_\alpha}{2}}\left|\mathbf{I}_{k-r_\alpha-2i-1}+\mathbf{T}_{22}^{r_\alpha+2i+1}\right|^{\frac{r_\alpha-1}{2}-\alpha}.$$

Indeed, if $\alpha\geq\frac{k-1}{2}$, then $r_\alpha=2\alpha$ and both of the convolutions are over an empty set of indices and $k-r_\alpha-1\leq 0$ is not a positive integer in which the case it is assumed that $\left|\mathbf{I}_{k-r_\alpha-1}+\mathbf{T}_{22}^{r_\alpha+1}\right|=1$, so both sides of the above equality are equal to one. Next, consider $\alpha<\frac{k-1}{2}$ and let $I_L=\lfloor\frac{k-r_\alpha}{2}-1\rfloor$ and $I_R=\lfloor\frac{k-r_\alpha-1}{2}\rfloor$. If $I_L=I_R$, then $\left|\mathbf{I}_{k-r_\alpha-2I_L-2}+\mathbf{T}_{22}^{r_\alpha+2I_L+2}\right|=1$, since $k-r_\alpha-2I_L-2\leq 0$, and thus both the sides have the same factors. Similarly, if $I_L=I_R-1$, then $\left|\mathbf{I}_{k-r_\alpha-2I_R-1}+\mathbf{T}_{22}^{r_\alpha+2I_R+1}\right|=1$ since $k-r_\alpha-2(I_L+1)-1\leq 0$ and thus the equality is shown. The equality of analogous terms corresponding to $\alpha+\beta$ and β follow by a similar argument.

□

Proof of Proposition 5.4. The process **GL** is the sum of k independent $k\times k$ processess $\mathbf{X}_l$, $l=1,\ldots,k$, such that for $\mathbf{X}_l=\mathbf{X}_l^\top$,

$$\mathbf{X}_{l;l\cdot}=\Big(0\ldots 0\,\Gamma_l(s_l)\,\tfrac{\sqrt{2}}{2}B_{l,l+1}(\Gamma_l(s_l))\ldots\tfrac{\sqrt{2}}{2}B_{l,k}(\Gamma_l(s_l))\Big),$$
$$\mathbf{X}_{l;\cdot l}=\mathbf{X}_{l;l\cdot}^\top,$$

and having all other entries equal to zero. The ChF of $\mathbf{X}_l = \mathbf{X}_l(\mathbf{s})$ is given by

$$\begin{aligned}
\phi_{\mathbf{X}_l}(\mathbf{T}) &= \mathbb{E}\left[e^{\iota t_{ll} G + \iota\sqrt{2}\tilde{\mathbf{t}}_l\sqrt{G}\mathbf{Z}}\right] \\
&= \mathbb{E}\left\{\mathbb{E}\left[e^{\iota t_{ll} G + \iota\sqrt{2}\tilde{\mathbf{t}}_l\sqrt{G}\mathbf{Z}} | G\right]\right\} \\
&= \int_0^\infty e^{\iota t_{ll} x}\, \mathbb{E}\left[e^{\iota\sqrt{2x}\tilde{\mathbf{t}}_l\mathbf{Z}}\right] \frac{x^{s_l-1}}{\Gamma(s_l)} e^{-x} dx \\
&= \int_0^\infty e^{\iota t_{ll} x}\, e^{-x\left(1+\tilde{\mathbf{t}}_l\tilde{\mathbf{t}}_l^\top\right)} \frac{x^{s_l-1}}{\Gamma(s_l)} dx \\
&= \frac{1}{\left(1+\tilde{\mathbf{t}}_l\tilde{\mathbf{t}}_l^\top\right)^{s_l}} \int_0^\infty e^{\iota t_{ll} x}\, e^{-x\left(1+\tilde{\mathbf{t}}_l\tilde{\mathbf{t}}_l^\top\right)} \frac{\left(1+\tilde{\mathbf{t}}_l\tilde{\mathbf{t}}_l^\top\right)^{s_l} x^{s_l-1}}{\Gamma(s_l)} dx \\
&= \frac{1}{\left(1+\tilde{\mathbf{t}}_l\tilde{\mathbf{t}}_l^\top\right)^{s_l}} \frac{1}{\left(1-\iota\frac{t_{ll}}{1+\tilde{\mathbf{t}}_l\tilde{\mathbf{t}}_l^\top}\right)^{s_l}} \\
&= \frac{1}{\left(1+\tilde{\mathbf{t}}_l\tilde{\mathbf{t}}_l^\top - \iota t_{ll}\right)^{s_l}},
\end{aligned}$$

where $G \stackrel{d}{=} \Gamma(s_l)$ and $\mathbf{Z}$ is a column of $k-l$ independent standard normal variables when $l<k$ and is zero when $l=k$. The obtained ChF is clearly infinitely divisible with the group parameter s_l. Consequently, the independent processes $\mathbf{X}_l(\mathbf{s})$ are Lévy motions and thus so is $\mathbf{GL} = \mathbf{X}_1 + \cdots + \mathbf{X}_k$. The proof for the LT is similar. □

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