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Abstract The paper focuses on the option price subdiffusive model under the unusual behavior of the market, when the price may not be changed for some time which is quite a common situation in the modern financial markets or during global crises. In the model, the risk-free bond motion and classical GBM are time-changed by an inverted inverse Gaussian (IG) subordinator. We explore the correlation structure of the subdiffusive GBM stock returns process, discuss option pricing techniques based on the fractal Dupire equation, and demonstrate how it applies in the case of the IG subordinator.

Keywords Option pricing, Subdiffusion models, Subordinator, Inverse subordinator, Time-changed process, Hitting time

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1 Introduction

Nowadays, modeling the financial market dynamics using diffusion processes has become an active research area in financial risk management, asset valuation, and derivatives pricing. However, the classical diffusion models like Black-Scholes-Merton (B-S) and others based on Brownian motion (BM) have

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a normal distribution, independence of returns and are perpetually moving. These assumptions are inconsistent with empirical properties such as heavy-tailed and skewed marginal distributions, dependence on squared returns, and, constant motionless periods (called also trapping events). The trapping events can be observed, for example, during global crises that negatively affect financial activity, and some types of risky assets have periods in their dynamics without changes. Such behavior too is typical for emerging markets with a low number of transactions, for interest rate markets, and for commodity markets.

To overcome these difficulties one can notice, the constant periods of stagnation in financial processes are analogous to the trapping events of the subdiffusive particle, therefore, the physical models of subdiffusion (a kind of anomalous diffusion, see [14]) can be successfully applied to describe financial data. Subdiffusion in the different physical systems arises from some memory effect of previous states, as a result of some fractal structure of the background space, or due to some non-linear interactions inherent in the system, etc. To model subdiffusion, the time structure of the stochastic process is changed, and the time-changed process is no longer Markovian Dynamics. (Markovian Dynamics means each new step in the motion depends only on the present state and is independent of the previous states.)

The idea of the time-changed process was introduced by [1]. The theory of subordinated processes is also explored in detail by [15].

The time change can be considered the replacement of calendar time in the considered process S_t using some subordinator G_t (non-decreasing process with stationary, independent increments). The inverse subordinator is used as a new random ("hitting") time H_t, thus as a studied model we obtain S_{H_t}. The distributional properties, asymptotic behavior, and the simulation procedures of the time-change process were considered for different types of subordinators and their inverse: α-stable, exponential, gamma, Pareto, Mittag-Leffler, and tempered distributions (see Cont and Tankov [4], Janczura *et al.* [8], Kumar *et al.* [10], Wylomańska *et al.* [19], Beghin *et al.* [2] and other).

However, the pricing of derivatives in this framework remains a complicated and under-researched problem. In the papers, [12, 13] applied the subdiffusive mechanism of trapping events to describe financial data demonstrating periods of constant values and introduced the subdiffusive geometric BM and arithmetic BM. Magdziarz showed that the considered models are arbitrage-free but incomplete, and obtained the corresponding subdiffusive formulas for the fair prices of European options. The other technique for European option pricing was proposed in Donatien and Leonenko [5]. In the PIDE the derivative in the direction time is replaced by a Dzerbayshan–Caputo (D–C) derivative. The theory in these papers was detailed for inverse α - stable [12, 13, 5], inverse tempered stable [13], and inverted Poisson processes [5].

In the paper, we consider the geometric BM model in a subdiffusion regime, time-changed by an Inverse Gaussian (IG) subordinator. We aim to show, that the studied subdiffusive model demonstrates long-range dependence of stock returns and discuss two different ways for option pricing. In addition, we propose a procedure for evaluating value-at-risk in the studied model. From this perspective, our study closes a gap that is of particular interest to investors.

The paper is organized as follows.

For introducing a subdiffusive model for the dynamics of a financial market we assume that the market consists of at least three components: one riskless asset B_t, one risky asset with price S_t, and one derivative security, usually called call option with price C_t. The dynamics of the first two components in the subdiffusive framework are presented in the second section. We consider the IG process as the subordinator and the IIG process as the inverse subordinator for subdiffusive GBM, describe their properties and features, demonstrate the simulation for them, and for the subdiffusive GBM stock returns processes. We explore the correlation structure of the subdiffusive GBM stock returns process and assume that the stock returns process has long-range dependence and it is presented in the squared returns. Finally, we mentioned the Fractional Fokker-Planck equation (FFPE) for IG subordinator as the usual approach for modeling sub-diffusion in physics. This equation describes the probability density function $w(t)$ of the sub-diffusive studied stock process and we discuss how it can be used for risk measuring.

The next section focuses on the third component of the model and discusses two option pricing techniques. The one technique is very common for option pricing and can be found as a discounted mathematical expectation of the option's payoff function. The other technique is based on the fractal Dupire equation. Since the form of the D–C derivative depends upon the chosen inverted Lévy subordinator, we demonstrate how it applies to the IG subordinator. The fourth section contains the numerical illustration for real financial data.

2 Subdiffusive GBM model with *IG* subordinator

To build the mathematical model for the dynamics of a financial market first, we need to assume what kinds of securities evolve on the market and to describe their dynamics. Assume that the market consists of at least one riskless asset, usually called bond B_t, a risky asset with price S_t, usually called the stock, and one derivative security, usually called call option, or put option, which will have a certain payoff at a specified date in the future, depending on the values taken by the stock up to that date.

The idea is to replace the calendar time t in risk-free bond motion and classical GBM by some stochastic process H_t, which means stochastic clock, operation time.

The time-changed risk-free bond has a value at time t equal to:

$$\frac{dB_{H_t}}{B_{H_t}} = r dH_t, B_0 = 1, \tag{1}$$

and the movement of the underlying risk assets S_t follows a subdiffusive geometric Brownian motion (GBM):

$$\frac{dS_{H_t}}{S_{H_t}} = \left(\mu + \frac{\sigma^2}{2}\right) dH_t + \sigma dB_{H_t}, \quad t > 0, \tag{2}$$

with solution

$$S_{H_t} = S_0 e^{\mu H_t + \sigma B_{H_t}}, \quad t > 0. \tag{3}$$

In formula (3) the standard diffusive process S_{H_t} is time-changed by some stochastic process H_t, which is called the inverse subordinator ("hitting time"). The inverse subordinator H_t is defined as

$$H_t = inf\left(\tau > 0 : G_\tau \geq t)\right), \tag{4}$$

and interpreted as the first time at which G_t hits the barrier t. Thus it is time of first reaching a certain price, which may not change for some time. The definition (4) of the inverse subordinator is based on the use of some other random process called a subordinator G_t.

The subordinator G_t is generally a non-decreasing stochastic process with stationary independent increments (Lévy process), taking value in $R+$ and having Laplace transform in the following form:

$$E\left(e^{-uZ_t}\right) = e^{-t\Psi(u)}, \tag{5}$$

where $\Psi(u)$ is called Lévy exponent, which can be written in the following form

$$\Psi(u) = bu + \int_0^{+\infty} \left(1 - e^{-ux}\right) \tilde{\nu}(dx). \tag{6}$$

Here, $b \geq 0$ is the drift parameter. If for simplicity, following [8], we assume $b = 0$, then $\tilde{\nu}(dx)$ is an appropriate Lévy measure.

The subordinator G_t in our framework is often called the "waiting" time. In this paper, we consider the Inverse Gaussian process as a subordinator.

2.1 IG subordinator and its inverse

Inverse Gaussian (IG) subordinator G_t is a nondecreasing Lévy process, where the increments $G_{t+s} - G_s$ follow the inverse Gaussian distribution $\varrho(\delta t, \gamma)$ with probabilities density function (PDF) [4]:

$$g(x,t) = \frac{\delta t}{\sqrt{2\pi x^3}} e^{\delta\gamma t - (\delta^2 t^2/x + \gamma^2 x)/2}, \quad x > 0;$$

and with Lévy measure

$$\tilde{\nu}(dx) = \frac{\delta}{\sqrt{2\pi x^3}} e^{\left(-\frac{\gamma^2 x}{2}\right)} dx, \quad x > 0, t > 0. \tag{7}$$

For $\gamma = \delta = 1$ we have standard IG distribution in the form

$$f(x,t) = \frac{t}{\sqrt{2\pi x^3}} e^{\left(-\frac{(x-t)^2}{2x}\right)}, \quad x > 0, t > 0.$$

If we find the Laplace transformation of IG Lévy subordinator

$$E\left(e^{-uG_t}\right) = e^{-t\delta(\sqrt{2u+\gamma^2} - \gamma)},$$

therefore, the Laplace exponent for IG process is given by

$$\Psi_{IG}(u) = \delta(\sqrt{2u+\gamma^2} - \gamma). \quad (8)$$

The tail probability for the IG subordinator is in the form [19]:

$$P(G_t > x) \sim \sqrt{\frac{2}{\pi}} \frac{\delta t}{\gamma^2} e^{\gamma \delta t} x^{-3/2} e^{-(\gamma^2/2)x}, \quad x \to \infty,$$

where $f(x) \sim g(x)$ as $x \to 0$ means $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$. Notice, that the tail probability decreases exponentially. Thus all moments of the $G(t)$ process are finite.

The qth moments for the IG process were found in paper Jørgensen [9]:

$$EG^q(t) = \sqrt{\frac{2}{\pi}\delta} \left(\frac{\delta}{\gamma}\right)^{q-1/2} t^{q+1/2} e^{\delta\gamma t} K_{q-1/2}(\delta\gamma t),$$

where $K_q(\omega)$ is the modified Bessel function of the third kind with index q. Moreover in Jørgensen [9] was shown that if $t \to \infty$, then

$$E(G^q(t)) \sim \left(\frac{\delta}{\gamma}\right)^q t^q. \quad (9)$$

For standard distribution, for any moment t we have $E(G(t)) \sim t$, $var(G(t)) \sim t^2$.

The algorithm of the simulation of the IG process $G(t)$ for time points $t_1 = \frac{1}{n}, t_2 = \frac{2}{n}, ..., t_n = 1$ was proposed in some literature (see for example Wylomańska *et al.* [19], Cont and Tankov [4]). Since the process $G(t)$ has independent and stationary increments $F_i = G(t_i) - G(t_{i-1}) = G(dt) \sim \varrho(dt, 1)$ for $i = 1, 2, ...n$ and $dt = \frac{1}{n}$, so we follow for [19] and generate n i.i.d IG variables F_i assuming $\gamma = \delta = 1$. For this, we generate a standard normal random variable N and a uniform $[0, 1]$ random variable U. Then assign $X = N^2$ and $Y = dt + \frac{X}{2} - \frac{1}{2}\sqrt{4dtX + X^2}$. According this algorithm if $U \le \frac{dt}{dt+Y}$ return Y; otherwise return $\frac{(dt)^2}{Y}$. After assigning $G(t_0) = 0$ and $G(t_i) = \sum_{j=1}^{i} F_j, i = 1, 2, ..., n$ we obtain $G(t_1), G(t_2), ..., G(t_n)$, which are simulated values of the IG process at times $t_1, t_2, ..., t_n$ respectively. The trajectory $G(t)$ simulation for the standard IG process is demonstrated below in Figure 1.

Now we discuss the properties of the inverse subordinator (hitting time) H_t defined by (4). The inverse to the inverse Gaussian (IIG) process is not Lévy and has monotonically increasing continuous sample paths. Moreover, the sample paths of the IIG process are constant over the intervals where $G(t)$ has jumped. It follows from fact, that the trajectories of $G(t)$ are strictly increasing with jumps.

The distribution of H_t is not infinitely divisible [19].

The density function $h(x, t)$ of H_t can be put in the following integral form [18]:

$$h(x,t) = \frac{\delta}{\pi} e^{\delta\gamma x - \frac{\gamma^2}{2}} \int_0^\infty \frac{e^{-ty}}{y + \frac{\gamma^2}{2}} (\gamma \sin(\delta x\sqrt{2y}) + \sqrt{2y}\cos(\delta x\sqrt{2y}))dy. \quad (10)$$

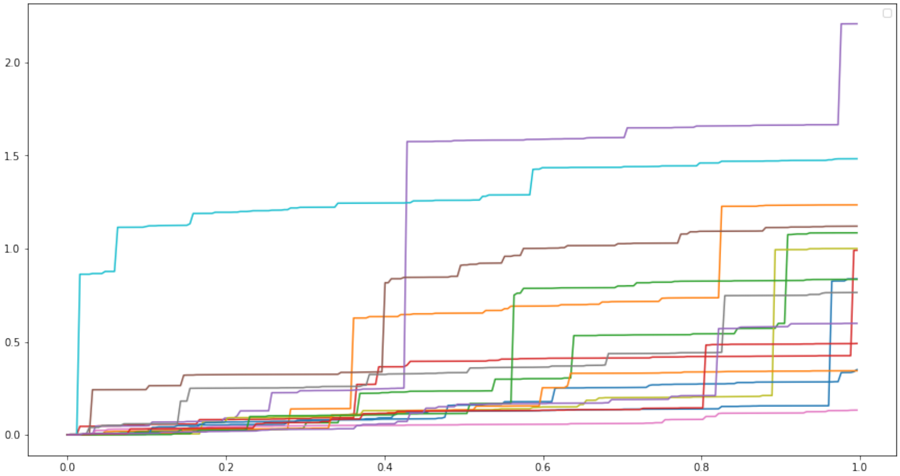

Fig. 1. Simulation of the IG process trajectories for $\gamma = \delta = 1$

For the particular case $\gamma = 0, \delta = 1$:

$$h(x,t) = \frac{\sqrt{2}}{\pi t} e^{-\frac{x^2}{2t}}.$$

The qth moments for the IIG process may be numerically evaluated for known t by using the density function $h(x,t)$. However, an explicit expression for the first and second order was obtained by using the Laplace transformation in Vellaisamy and Kumar [18].

Moreover, in Jørgensen [9] you can find the following result regarding the asymptotic behavior of $E(H_t)$ and $Var(H_t)$. If $t \to \infty$,then

$$E(H(t)) \sim \begin{cases} \left(\frac{\gamma}{\delta}\right) t, & \gamma > 0 \\ \left(\frac{1}{\delta}\sqrt{\frac{2t}{\pi}}\right) t, & \gamma = 0. \end{cases} \tag{11}$$

$$Var(H(t)) \sim \left(\frac{\gamma}{\delta}\right)^2 t^2 \tag{12}$$

For standard distribution, for $\gamma = \delta = 1$ we have $E(H(t)) \sim t$, $var(H(t)) \sim t^2$.

To simulate the approximate trajectory of the inverse subordinator H_t, we define $H_\Delta(t))$ with the step length δ as follows [19]:

$$H_\Delta(t) = [\min\{n \in N : G(\Delta n) > t\} - 1]\Delta, \quad n = 1, 2, \ldots \tag{13}$$

where Δ is the step length and $G(\Delta n)$ is the value of the Inverse Gaussian process G_t evaluated at n. The simulation of the trajectory H_t for $\gamma = \delta = 1$ is demonstrated in Figure 2.

For modeling stochastic subdiffusive GBM we propose the next iterative scheme

$$x_{k+1} = x_k + \mu x_k H_\Delta(t) + \sigma x_k \sqrt{H_\Delta(t)} \varepsilon_k, \tag{14}$$

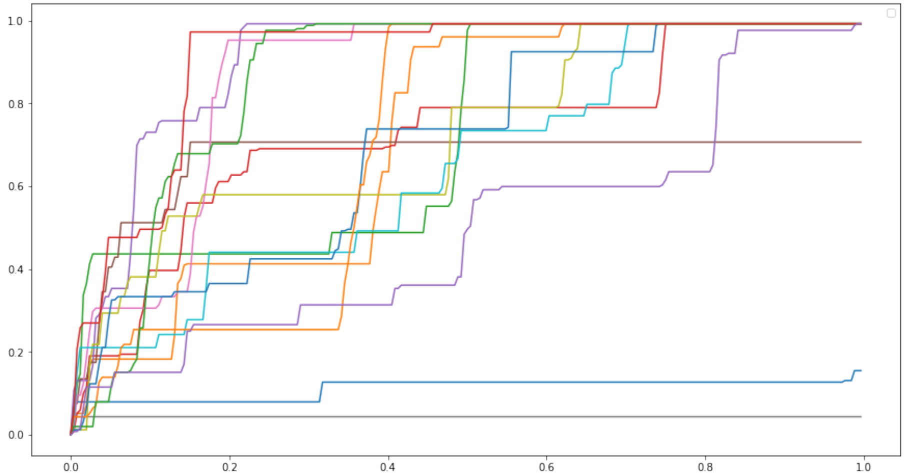

Fig. 2. Simulation of the inverse to the IG process trajectories for $\gamma = \delta = 1$

where ε is white noise with normal standard distribution, $H_\Delta(t)$ follow (13). The trajectories for the subdiffusion GBM with the inverse to the IG process is demonstrated in Figure 3.

2.2 *Correlation structure for log returns of time-changed GBM*

Exploring the dependence structure for stock processes or its stock returns is an important issue in the study of diffusive and subdiffusive models (see for example Wylomańska *et al.* [19], Kumar *et al.* [10], Casteli *et al.* [3]).

First, we compute

$$logS_{H_t} = logX_0 + \mu H_t + \sigma B^{(1)}_{H_t};$$

where S_{H_t} is a stochastic process given by (3).

Then denote $\tau_t = H_t - H_{t-1}$ and obtain the stock returns in the form:

$$X(t) = X_{H_t} = log\frac{X_{H_t}}{X_{H_{t-1}}} = \mu\tau_t + \sigma\sqrt{\tau_t}B^{(1)}_1; \tag{15}$$

using the scaling law of Brownian motion.

Proposition 1. *Let $X(t)$ is a stochastic process given by (15), then for any integer $k \geq 0$:*

1. *Stock returns are uncorrelated, for $\mu = 0$:*

$$\mathrm{Cov}(X(t), X(t+k)) = \mathrm{Cov}(X_{H_t}, X_{H_{t+k}}) = \mu^2\mathrm{Cov}(\tau_t, \tau_{t+k}); \tag{16}$$

in particular $\mathrm{Cov}_{\mu=0}(X_t, X_{t+k}) = 0.$

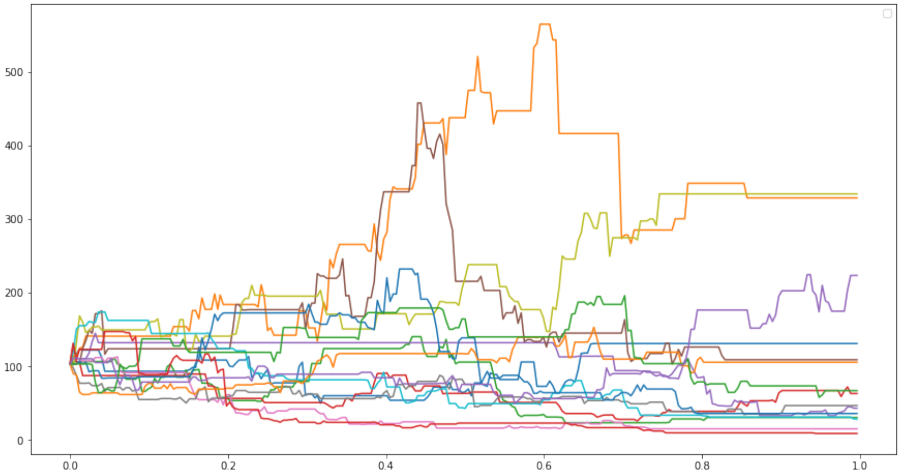

Fig. 3. Simulation of the subdiffusive GBM time-changed by inverse IG subordinator

2. There is long-range dependence in the squared returns:

$$\mathrm{Cov}(X^2(t), X^2(t+k)) = \mathrm{Cov}(X^2_{H_t}, X^2_{H_{t+k}}) =$$

$$\sigma^4\mathrm{Cov}(\tau_t, \tau_{t+k}) + \mu^4\mathrm{Cov}(\tau_t^2, \tau_{t+k}^2) + 2\mu^2\sigma^2\mathrm{Cov}(\tau_t^2, \tau_{t+k}); \qquad (17)$$

in particular $\mathrm{Cov}_{\mu=0}(X^2_{H_t}, X^2_{H_{t+k}}) = \sigma^4\mathrm{Cov}(\tau_t, \tau_{t+k})$.

Proof. For the covariance function of the log returns process $X(t)$ (15), we obtain:

$$\mathrm{Cov}(X_{H_t}, X_{H_{t+k}}) = \mathrm{Cov}(\mu\tau_t + \sigma\sqrt{\tau_t}B_1^{(1)}, \mu\tau_{t+k}\sigma\sqrt{\tau_{t+k}}B_1^{(2)}) =$$

$$\mathrm{E}[(\mu\tau_t + \sigma\sqrt{\tau_t}B_1^{(1)} - \mu E(\tau_t)(\mu\tau_{t+k} + \sigma\sqrt{\tau_{t+k}}B_1^{(2)} - \mu E(\tau_{t+k})] =$$
$$= \mu^2(\mathrm{E}[\tau_t\tau_{t+k}] - \mathrm{E}[\tau_t]\mathrm{E}[\tau_{t+k}]) = \mu^2\mathrm{Cov}(\tau_t, \tau_{t+k}),$$

where the second equation follows from the fact that $\mathrm{E}[X_t] = \mu\mathrm{E}[\tau_t]$, and the third from the independence between the two stochastic processes.

The covariance function of the process $X^2(t)$ can be computed as

$$\mathrm{Cov}(X_t^2, X_{t+k}^2) = \mathrm{Cov}((\mu\tau_t + \sigma\sqrt{\tau_t}B_1^{(1)})^2, (\mu\tau_{t+k} + \sigma\sqrt{\tau_{t+k}}B_1^{(2)})^2) =$$

$$= \sigma^4\mathrm{Cov}(\tau_t, \tau_{t+k}) + \mu^4\mathrm{Cov}(\tau_t^2, \tau_{t+k}^2) + \mu^2\sigma^2\mathrm{Cov}(\tau_t^2, \tau_{t+k}) + \mu^2\sigma^2\mathrm{Cov}(\tau_{t+k}, \tau_t^2),$$

where from the last expression we immediately get the result. □

Thus, if the subordinated process H_t has long-range dependence of its increments τ_t, then the same structure is present in the log returns of the process $X^2(t)$ and we have long-range dependence in the squared returns.

2.3 Fractional Fokker-Planck equation and risk measuring for subdiffusion

The usual model of subdiffusion in physics is the celebrated Fractional Fokker-Planck equation (see for example [14]). This equation was derived from the continuous-time random walk scheme with heavy-tailed waiting times and describes the probability density function $w(t)$ of the sub-diffusive studied stock process:

$$\frac{\partial w}{\partial t} = \Phi_t[-\mu \frac{\partial}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2}] w(x,t), \tag{18}$$

where Φ_t is the integro-differential operator defined as

$$\Phi_t f(t) = \frac{d}{dt} \int_0^t M(t-y) f(y) dy,$$

with the memory kernel $M(t)$ defined via its Laplace transform

$$\tilde{M}(u) = \int_0^\infty e^{-ut} M(t) dt = \frac{1}{\Psi(u)},$$

Levy exponent $\Psi(u)$ for IG subordinator can be written as (8), what implicates that the memory kernel $M(t)$ can be expressed as:

$$M(t) = L^{-1} \left(\frac{1}{\delta(\sqrt{2u + \gamma^2} - \gamma)} \right). \tag{19}$$

where $L^{-1}(f)$ is the inverse Laplace transform of the $f(t)$ function. Thus, the formula (18) allows us to find, at least in some particular cases of parameters γ, δ closed-form formulas for the PDF of the sub-diffusive studied stock process. In general, approximated solutions $w(t)$ of (18) can be derived by the finite element method for FFPE (see for example Deng [6]) or by the Monte Carlo techniques based on the simulation algorithm of the time-changed stock process (see the section above).

Thus, the possibility of numerical computing probability density function $w(t)$ for the sub-diffusive studied stock process (with IG subordinator) opens the way to evaluate value-at-risk (VaR) in this model.

The value-at-risk is a quite useful tool for investors and can be used for understanding the past and making medium-term and strategic decisions for the future. On the other side, we can apply VaR for the checking model performance. For this, we can use the most important criterion of a risk management system, namely to check if the regulatory requirements are fulfilled.

VaR can be defined as α-quantile of the profit (loss) function.

Let $(\Omega, \mathcal{F}, P)$ be the probability space. The value-at-risk of level α, $0 < \alpha \leq 1$ is a probability functional, defined as α-quantile of the profit (loss) function $Y \in L(\Omega)$:

$$VaR_\alpha(Y) = W^{-1}(\alpha) = inf\{ \ y \in R : \alpha \leq W(Y)\} \ , \tag{20}$$

where W is the distribution function of Y, W^{-1} is the quantile function of α, $0 < \alpha \leq 1$.

Let the time horizon coincide with the time to maturity, then the loss(profit) function of the call option with strike price K is

$$Y = Y(S) = |S - K|^{+} - c_0,$$

Then the value-at-risk of level α, $0 < \alpha \leq 1$ for random variable Y is

$$VaR_\alpha(Y) = VaR_\alpha(|S - K|^{+}) - c_0 \tag{21}$$

due to the translation-equivariant property of probability functional VaR.

The cumulative distribution function (CDF) for $Y(S) = |S - K|^{+}$ is given [16]:

$$W_Y(y) = \begin{cases} \int\limits_{-\infty}^{y} w_S(u + K)\, du, & y \geq 0 \\ 0, & y < 0. \end{cases} \tag{22}$$

Thus, if the time horizon coincides with the time to maturity, the value-at-risk of level α, $0 < \alpha \leq 1$ one can find as (21-22), where $w(t)$ is a solution of (18) with kernel memory (19).

3 Option pricing

Even if sub-diffusions can be successfully applied for modeling illiquidity, options pricing in this framework remains under-researching.

Recall that in the classical GBM model the fair price of the European call option is given by the Black-Scholes formula:

$$C(S, K, T, r, \sigma) = N(d_1)S - N(d_2)Ke^{-rT}, \tag{23}$$

where

$$d_1 = \frac{\log \frac{S_0}{K} + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \; d_2 = \frac{\log \frac{S_0}{K} + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \tag{24}$$

are both functions of five parameters: T, K, S_0, r, σ, and $\Phi(\cdot)$ is a standard normal cumulative distribution function.

Consider a time-changed version of the B-S model, where the price of the bond evolves as (1) and the underlying risky assets follow (2) with IG subordinator. Now we discuss two approaches to option pricing in the proposed model with inverse Gaussian subordinator.

Let the evolution of this market up to time horizon T be contained in the probability space $(\Omega, \mathcal{F}, P)$. Here, Ω is the sample space, $\mathcal{F}$ contains all statements, that can be made about the behavior of prices, and P is the "objective" probability measure. We denote by $(\mathcal{F}_t), t \in [0, T]$ the information about the history of asset prices S_H up to time t. $(\mathcal{F}_t)$ is also called filtration and is interpreted as the background information that is available for the investor. The more time proceeds the more information is revealed to the investor.

Proposition 2. *The European call option price $C_H = C_H(S, K, T, \sigma)$ for a time-changed version of the B-S model with IG subordinator satisfies*

$$C_H = \int_0^\infty \int_0^\infty C(S, K, x, \sigma) \frac{\delta}{\pi} e^{\delta\gamma x - \frac{\gamma^2}{2}} \frac{e^{-Ty}}{y + \frac{\gamma^2}{2}} (\gamma \sin(\rho(x)) + \sqrt{2y} \cos(\rho(x))) dy dx, \tag{25}$$

where

$$\rho(x) = \delta x \sqrt{2y}, \tag{26}$$

$C(S, K, T, \sigma)$ is given by (23).

Proof. For the sub-diffusion market described above the usual requirement for fair option pricing is that arbitrage opportunities do not exist. For this, it is enough to prove the existence of the equivalent martingale measure. In [12] was introduced the following measure

$$\mathbb{Q}(A) = \int_A \exp\left\{-\gamma B(S_{H_T}) - \frac{\gamma^2}{2} S_{H_T}\right\} d\mathbb{P}, \tag{27}$$

where $\gamma = \frac{\mu}{\sigma}$ and $A \in \mathcal{F}$, and was proved that (2) is a martingale with respect to Q. Also in [12] was shown that the market model, in which the asset price is described by (2), is arbitrage-free. The second question is the completeness of the market model. The Second Fundamental theorem of asset pricing states that the model is complete if and only if there is a unique martingale measure. Then, for arbitrage-free and incomplete markets, we apply a common technique for time-changed processes (see [12],[13], [3], [7]) and find the corresponding fair price of the European call option written on the time-changed asset as the discounted expected payoff under measure Q (27):

$$C_H(S, K, T, \sigma) = \langle C(S, K, H(T), \sigma) \rangle = E^Q \left(e^{-rH(T)} (S_H - K)^+ \middle| \mathcal{F}_0 \right).$$

Therefore, conditioning on H_T, we obtain

$$C_H(S, K, T, \sigma) = \int_0^\infty C(S, K, x, \sigma) h_\Psi(x, T) dx \tag{28}$$

Here, $h_\Psi(x, T)$ is the PDF of $H(T)$ for the subordinator with Levi exponent Ψ and $C(S, K, T, \sigma)$ is given by (23).

If we note that $h_\Psi(x, T)$ for the considered model is PDF of IGG process (10) then this ends the proof of formula (25). □

Remark 1. *In the above equation (28) we can evaluate the subdiffusive call price $C(\cdot)$ by computing the integral numerically. An alternative consists of calculating the price by Monte Carlo simulations. The first one simulates the trajectories for the inverse subordinator on the interval $[0, T]$ by the approximation scheme (13). Then, one obtains the fair price as an estimation of the expected value for simulated prices where the inverse subordinator stands for calendar time T in (28)*

$$C_H(S, K, T, r, \sigma) = \langle C(S, K, H(T), \sigma) \rangle = \frac{1}{n} \sum_{i=1}^{n} C(S, K, H_i(T), \sigma), \tag{29}$$

where $C(S, K, T, \sigma)$ is taken from Black-Scholes option pricing formula (23).

It is worth noticing, that the application of the Monte-Carlo method for option pricing in subdiffusive models can be seen in the papers [12], [13] for α stable subordinator.

The second approach to option pricing in the considered model is much more interesting and based on a fractional version of what is called Dupire's equation. Dupire has established a forward partial differential equation for call options with local volatility. The fractional Dupire's equation (PIDE) was proposed by [5]. This equation was presented in a very general form and valid for all invertible Lévy subordinators. In PIDE the derivative with respect to time was replaced by a convolution-type derivative, called Dzerbayshan–Caputo (D–C) derivative. (D–C) derivative is a kind of fractional derivative, which is more advantageous than its classical counterparts due to capturing the past history. The Dzerbayshan–Caputo (D–C) derivative depends upon the chosen kind of subordinator and Bernstein functions $f(.)$.

The next proposition is the application of this PIDE to B-S subdiffusion with IG subordinators.

Proposition 3. *The European call option price $C_H = C_H(S, K, T, \sigma)$ for a time-changed version of the B-S model with IG subordinator is a solution of a fractional PIDE equation:*

$$ {}^{f}DC_H(T,k) = -r\frac{\partial}{\partial k}C_H(T,k) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial k^2}C_H(T,k), \tag{30} $$

where $k = lnK$ and

$$ {}^{f}Du(t) = \delta \int_0^t \frac{\partial}{\partial t}u(t-s)\left(2\Phi\left(\frac{\sqrt{2s}}{2}\right) - 1 + \frac{2e^{-\frac{s}{2}}}{\sqrt{2\pi s}}\right)ds \tag{31} $$

is the convolution-type derivative, called the Dzerbayshan–Caputo (D–C) derivative.

Proof. In [5] was presented a fractional PIDE equation for option pricing in the fractional jump–diffusion setting. From this PIDE in the Black and Scholes (B-S) regime, when the Brownian volatility is constant and there are no jumps, the fraction Dupire PIDE can be rewritten as (30). The generalized D–C derivative according to the function f is defined as [5]:

$$ {}^{f}Du(t) = b\frac{d}{dt}u(t) + \int_0^t \frac{\partial}{\partial t}u(t-s)\nu(s)ds, \tag{32} $$

where f is Bernstein function (see for the details [17]), which admits a similar representation to the Laplace exponent of Lévy process

$$ f(x) = a + bx + \int_0^{+\infty}(1-e^{-sz})\tilde{\nu}(dz), $$

and $(a, b, \tilde{\nu})$ is the Lévy triplet of the Bernstein function.

We can notice, that the term of Bernstein functions goes back to the potential theory school of A. Beurling and J. Deny and was subsequently adopted by C. Berg and G. Forst. S. Bochner calls them completely monotone mappings and probabilists still prefer the term Laplace exponents.

The Dzerbayshan–Caputo (D–C) derivative depends upon the chosen Bernstein function $f(.)$. For IG Lévy subordinator the Lévy triplet of the Bernstein function is $(0, 0, \tilde{\nu})$. The Lévy measure $\tilde{\nu}(s)$ is defined by (7). The tail of the Lévy measure is given by

$$\nu(s)ds = \left(a + \int_s^{+\infty} \tilde{\nu}(dx)\right) ds = ds \int_s^{+\infty} \tilde{\nu} dx = ds \int_s^{+\infty} \frac{\delta}{\sqrt{2\pi x^3}} e^{\left(-\frac{\gamma^2 x}{2}\right)} dx \tag{33}$$

If we compute the integral in (33), D–C derivative (32) for IG subordinator can be written as

$$^f Du(t) = \delta \int_0^t \frac{\partial}{\partial t} u(t-s) \left(erf\left(\frac{\sqrt{2s}}{2}\right) - 1 + \frac{2e^{-\frac{s}{2}}}{\sqrt{2\pi s}} \right) ds, \tag{34}$$

where $erf(.)$ is the error function, which can be expressed in terms of the standard normal cumulative distribution function $\Phi(\cdot)$. Thus, the proposition follows. □

4 Numerical illustration

We illustrate call option pricing for the subdiffusive model in the Black-Scholes regime with IG subordinator.

We choose Airbnb, Inc. Class A Common Stock listed on the NASDAQ stock exchange and use their daily returns during the last two years for model calibration. All the prices and relative data we take on 24 June 2022. We fixed the strike price as 100 and observed the prices of options with time to maturity ranging from the 1st of July 2022 to the 20th of January 2023. Market parameters are: $S_0 = 103.51$, $K = 100$, $r = 0.168$, $\mu = -0.0015$.

To simplify the calculations, we put parameters $\gamma = \delta = 1$ because this assumption allows getting the desired result regarding the asymptotic behavior of the hitting time. In this case when $t \to \infty$ it follows $E(H(t)) \sim t$, $var(H(t)) \sim t^2$. So, we estimate only the σ parameters based on the least squares technique. We obtain that the value $\sigma = 0.3$ is the best to minimize the square error over the difference between the real option quotes and the estimated ones.

We estimated the prices (29) of call options using Monte Carlo methods based on the above-described simulation procedure for H_T.

The results are presented in the table and in graphic shape, where we compare the B-S subdiffusive European call options with the classical one and with the market price.

As we can see from the graphics in Fig. 4, the diffusive option pricing model shows better results in the short-term period, while the subdiffusive model is more effective in the long-term perspective.

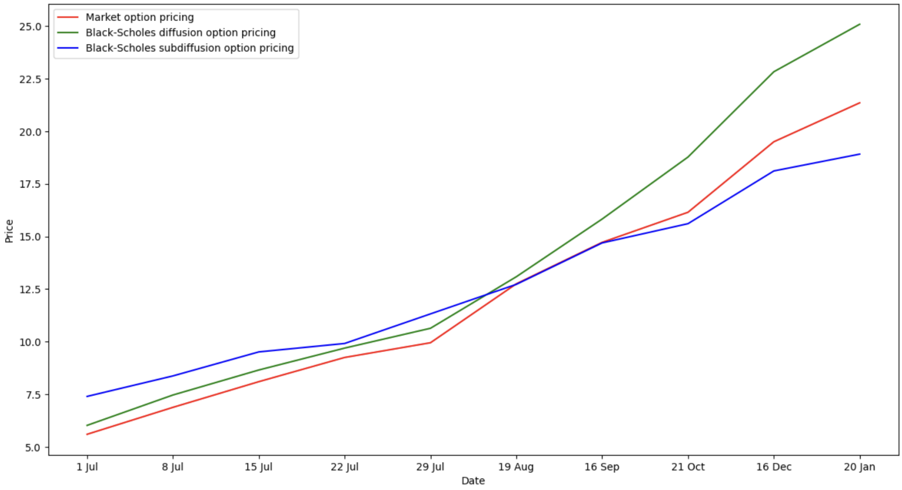

Fig. 4. Simulated prices for the diffusive and subdiffusive B-S models

Table 1. Call option prices for strike price K=100 and varying times of maturity

	1 Jul	8 Jul	15 Jul	22 Jul	29 Jul	19 Aug	16 Sep	21 Oct	16 Dec	20 Jan
Market	5.60	6.88	8.10	9.25	9.95	12.75	14.72	16.15	19.50	21.30
B-S diff	6.02	7.47	8.65	9.70	10.64	13.09	15.83	18.77	22.83	25.08
B-S subdiff	7.40	8.37	9.51	9.91	11.32	12.72	14.69	15.61	18.11	18.91

To compare numerical results we use absolute relative percentage (ARPE) and root mean squared error (RMSE):

$$ARPE = \frac{|C(T_k) - C_{market}(T_k)|}{C_{market}(T_k)} \tag{35}$$

$$RMSE = \sqrt{\frac{1}{n}\Sigma_{i=1}^{n}\Big(\frac{C(T_i) - C_{market}(T_i)}{C_{market}(T_i)}\Big)^2} \tag{36}$$

It is worth mentioning, that in econometrics, the root mean squared error (RMSE) (35) is a key criterion for model selection. The mean squared error indicates the mean squared deviation between the forecast and the outcome. It sums the squared bias and the variance of the estimator. The advantage of the ARPE (36) relative to the RMSE measure is that it gives a percentage value of the pricing error. Therefore, if we use both these errors it provides more insight into the economic significance of performance differences.

Upon dividing the RMSE analysis into two distinct periods, a noteworthy observation emerges: during the initial short period, the diffusive model outperforms, while in the subsequent long period, the subdiffusive model exhibits superior performance.

In the framework of the paper, we just illustrate the application of the studied model and compare option pricing results in a situation when strike

Table 2. The ARP errors for B-S diffusion, B-S subdiffusion to the market price.

	1 Jul	8 Jul	15 Jul	22 Jul	29 Jul	19 Aug	16 Sep	21 Oct	16 Dec	20 Jan	Mean
B-S	0.08	0.09	0.07	0.05	0.07	0.03	0.08	0.16	0.17	0.18	0.10
B-S Subdiffusion	0.32	0.22	0.17	0.07	0.14	0.00	0.00	0.03	0.07	0.11	0.11

Table 3. The RMS errors for diffusion and subdiffusion regarding the market price.

	1 Jul - 29 Jul	19 Aug - 20 Jan	Overall
B-S	0.07	0.14	0.11
B-S Subdiffusion	0.20	0.06	0.15

price K was fixed (in the money), while time to maturity T was changing.

For detailed model performance we need to examine the ARP pricing errors of the proposed option pricing models in more detail (see paper Lehar *et al.* [11]) and consider the pricing errors as a regression on the time to maturity T (in years), the moneyness of the option, and a binary variable that is set to unity, if the option is a call and to zero in the case of a put. This can indicate a level of explanatory value of moneyness, maturity, and the put-call dummy in the model.

However this model performance and application of a finite difference approach for solving the fractional Dupire equation is future work beyond the scope of the present paper.

5 Conclusion

The paper developed a sub-diffusive model with the following features of stock returns, which are quite well documented in the financial and econometric literature: i) the stochastic processes have continuous paths with motionless periods of time; ii) the returns processes are uncorrelated; iii) dependence is presented in squared returns; iv); the hitting time is defined by inverse to IG subordinator which is not infinitely divisible. For the model, two option pricing techniques were discussed. The results of comparing the studied model with the classical one show that the classical B-S model demonstrates better results in the short-term period, while the sub-diffusive model is more effective in the long-term perspective.

Thanks to the proposed approaches, the investor gets tools, that allow him to take into account the market's illiquidity.

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