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Abstract

This paper derives the density function of the product of a Wishart matrix and a normal vector that are independently distributed. Unlike existing results, we allow the covariance matrix of the Wishart distribution and that of the normal vector to differ, and we do not assume them to be proportional. In this general setting, the density is shown to admit a two-dimensional integral representation. We further obtain the density of linear combinations of the product. For a single linear combination, it is again expressed as a two-dimensional integral whose integrand involves the modified Bessel function of the second kind, and the special case of identity covariance matrices is treated explicitly. All resulting expressions are straightforward to evaluate numerically, and the product defines a new family of matrix-variate mixtures of multivariate normal distributions.

Keywords: matrix mixture of distributions, stochastic representation, Wishart distribution, inverse Wishart distribution, multivariate normal distribution, modified Bessel function of the second kind.

1 Introduction

Both the multivariate normal distribution and the Wishart distribution are fundamental in statistics and are widely used when statistical methods are applied to real data. Given a sample of independent observations from a multivariate normal distribution, the sample mean vector is multivariate normally distributed, the sample covariance matrix follows a Wishart distribution, and the two estimators are independent. These properties have a variety of uses in practice: the independence of the sample mean and covariance matrix is itself a characterizing property of the multivariate normal distribution (see, e.g., Lukacs [1942]), and it is the starting point for many theoretical results, for example in portfolio theory (see, e.g., Jobson and Korkie [1980], Kan and Zhou [2007], Kan and Smith [2008], Bodnar and Schmid [2009], Bodnar et al. [2016]).

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In many applications the Wishart and normal distributions appear jointly. The discriminant function (Muirhead [1982], Bodnar et al. [2020]) and the estimated mean-variance portfolio weights (Kan et al. [2022], Bodnar et al. [2023]), for example, involve the product of an inverse Wishart matrix and a normal vector (Bodnar and Okhrin [2011], Bodnar et al. [2022], Kan et al. [2024]). The product of a (non-inverse) Wishart matrix and a normal vector, which is the object of this paper, arises just as naturally – for instance when a sample covariance matrix acts on an independent normal vector – and its exact and approximate densities were studied by Bodnar et al. [2013], Bodnar et al. [2015], and Yonenaga and Suzukawa [2023]. Recently, Yonenaga [2026] extended these results to the product of a Wishart matrix and an elliptically distributed random vector.

From a probabilistic viewpoint, the product of a Wishart matrix and a normal vector can be regarded as a new family of matrix-variate mixtures of normal distributions, obtained by replacing the univariate generating variable of a classical normal mixture with a random matrix. Normal mixtures based on a univariate generating variable have been studied extensively and underlie several well-known families of multivariate distributions, including elliptically contoured distributions (Gupta et al. [2013]), skew-normal distributions (Azzalini [2005], Azzalini [2013]), and the Laplace distribution (Kotz et al. [2012], Kozubowski et al. [2025]). By contrast, the distributional properties of matrix-variate mixtures have received far less attention; some families of matrix-variate elliptically contoured distributions based on such mixtures are discussed in Fang and Zhang [1990].

This paper contributes to the statistical literature in two directions. First, we derive an analytical expression for the density function of the product of a Wishart matrix and a normal vector, extending Bodnar et al. [2013] and Yonenaga and Suzukawa [2023], who provide exact and approximate densities for linear combinations of the product. In particular, we treat the general case in which the covariance matrices of the Wishart and multivariate normal distributions need not be proportional. This substantially extends the framework of Bodnar et al. [2013], where proportional covariance matrices are assumed, while encompassing the more general covariance structure studied by Yonenaga and Suzukawa [2023]. The resulting density is expressed as a two-dimensional integral that is easy to implement in practice. The developed approach allows us to derive the joint density of all elements of the product, while the findings of Bodnar et al. [2013] and Yonenaga and Suzukawa [2023] are restricted to linear combinations of the product whose number is strictly smaller than the product dimension. Moreover, for the linear combinations our representations are one dimension lower than those of Bodnar et al. [2013] and Yonenaga and Suzukawa [2023], because the integral over the χ^2-distributed component is evaluated in closed form and yields a modified Bessel function of the second kind; the resulting expressions are therefore easier to evaluate numerically. More importantly, neither of these works delivers the joint density of the full product of a Wishart matrix and a normal vector, which corresponds to the case when the number of linear combinations is equal to the number of elements in the product and is the main object of the present paper. Second, we introduce a new class of multivariate distributions based on a matrix-variate mixture of multivariate normal distributions, extending the existing families of non-normal distributions.

The rest of the paper is organized as follows. The main results are presented in Section 2, where we derive the density of the product of a Wishart matrix and a normal vector. The density of linear combinations of the product, together with several important special cases, is given in Section 3. Concluding remarks are collected in Section 4, and some technical lemmas are deferred to the appendix (Section 5).

2 Distribution of the Product of the Wishart matrix and normally distributed random vector

Suppose $\mathbf{W} \sim W_m(n, \mathbf{\Sigma}_w)$ (m-dimensional Wishart distribution with n degrees of freedom and covariance matrix $\mathbf{\Sigma}_w$), $\mathbf{z} \sim N_m(\boldsymbol{\mu}, \mathbf{\Sigma}_z)$, and they are independent of each other. Throughout the paper, $\mathbf{\Sigma}_w$ and $\mathbf{\Sigma}_z$ are assumed to be positive definite, while the condition $n > m-1$ is imposed to guarantee the non-singularity of $\mathbf{W}$. Following Muirhead [1982, p.87], the degrees of freedom n are considered to be a real number throughout the paper.

In this section, we derive the density of $\mathbf{q} = \mathbf{Wz}$. This finding extends the results of Bodnar et al. [2013] and Yonenaga and Suzukawa [2023] who express the density of $\mathbf{L}'\mathbf{Wz}$ with $\mathbf{L}$ an $m \times p$ deterministic matrix as a $(p+2)$-dimensional integral. However, the matrix $\mathbf{L}$ must have rank $p < m$ in their setup and, as a result, these findings cannot be used to obtain the density of $\mathbf{q}$. In this section, we show that the density of $\mathbf{q}$ can be expressed as at most a double integral.

As an important special case, we consider $m = 1$, $n > 0$ and let $\mathbf{\Sigma}_w = \sigma_w^2$ and $\mathbf{\Sigma}_z = \sigma_z^2$. Let the symbol $\overset{d}{=}$ denote the equality in distribution. Then, $w \overset{d}{=} \sigma_w^2 v$ with $v \sim \chi_n^2$ and the density of $q = wz$ with $z \sim N(\mu, \sigma_z^2)$ is expressed as

$$f(q) = \mathbb{E}[f(q|v)] = \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_z \sigma_w^2 v} \exp\left(-\frac{(q - \sigma_w^2 v\mu)^2}{2\sigma_z^2 \sigma_w^4 v^2}\right) f_n(v) \mathrm{d}v, \tag{1}$$

where $f_n(\cdot)$ stands for the density function of a χ_n^2 random variable. This shows that when $m = 1$, the density function of q can be obtained with a single integral.

Next, we generalize (1) when $m \geq 2$ and $n > m-1$. Let $\tilde{\mathbf{W}} = \mathbf{\Sigma}_w^{-\frac{1}{2}} \mathbf{W} \mathbf{\Sigma}_w^{-\frac{1}{2}} \sim W_m(n, \mathbf{I}_m)$. Then, conditionally on $\tilde{\mathbf{W}}$, we have

$$\mathbf{q} = \mathbf{\Sigma}_w^{\frac{1}{2}} \tilde{\mathbf{W}} \mathbf{\Sigma}_w^{\frac{1}{2}} \mathbf{z} | \tilde{\mathbf{W}} \sim N(\mathbf{\Sigma}_w^{\frac{1}{2}} \tilde{\mathbf{W}} \mathbf{\Sigma}_w^{\frac{1}{2}} \boldsymbol{\mu}, \mathbf{\Sigma}_w^{\frac{1}{2}} \tilde{\mathbf{W}} \mathbf{\Sigma}_w^{\frac{1}{2}} \mathbf{\Sigma}_z \mathbf{\Sigma}_w^{\frac{1}{2}} \tilde{\mathbf{W}} \mathbf{\Sigma}_w^{\frac{1}{2}}), \tag{2}$$

and the conditional density of $\mathbf{q}$ is given by

$$f(\mathbf{q}|\tilde{\mathbf{W}}) = \frac{e^{-\frac{\boldsymbol{\mu}' \mathbf{\Sigma}_z^{-1} \boldsymbol{\mu}}{2}}}{(2\pi)^{\frac{m}{2}} |\mathbf{\Sigma}_w| |\mathbf{\Sigma}_z|^{\frac{1}{2}} |\tilde{\mathbf{W}}|} \exp\left(-\frac{\alpha_1}{2} + \alpha_2\right), \tag{3}$$

with

$$\alpha_1 = \mathbf{q}' \mathbf{\Sigma}_w^{-\frac{1}{2}} \tilde{\mathbf{W}}^{-1} \mathbf{\Sigma}_w^{-\frac{1}{2}} \mathbf{\Sigma}_z^{-1} \mathbf{\Sigma}_w^{-\frac{1}{2}} \tilde{\mathbf{W}}^{-1} \mathbf{\Sigma}_w^{-\frac{1}{2}} \mathbf{q} = \hat{\mathbf{q}}' \tilde{\mathbf{W}}^{-1} \tilde{\mathbf{\Sigma}}^{-1} \tilde{\mathbf{W}}^{-1} \hat{\mathbf{q}}, \tag{4}$$

$$\alpha_2 = \mathbf{q}' \mathbf{\Sigma}_w^{-\frac{1}{2}} \tilde{\mathbf{W}}^{-1} \mathbf{\Sigma}_w^{-\frac{1}{2}} \mathbf{\Sigma}_z^{-1} \boldsymbol{\mu} = \hat{\mathbf{q}}' \tilde{\mathbf{W}}^{-1} \hat{\boldsymbol{\mu}}, \tag{5}$$

where $\hat{\boldsymbol{\mu}} = \mathbf{\Sigma}_w^{-\frac{1}{2}} \mathbf{\Sigma}_z^{-1} \boldsymbol{\mu}$, $\hat{\mathbf{q}} = \mathbf{\Sigma}_w^{-\frac{1}{2}} \mathbf{q}$, and $\tilde{\mathbf{\Sigma}} = \mathbf{\Sigma}_w^{\frac{1}{2}} \mathbf{\Sigma}_z \mathbf{\Sigma}_w^{\frac{1}{2}}$. Equation (3) shows that the density $f(\mathbf{q}|\tilde{\mathbf{W}})$ depends on $\tilde{\mathbf{W}}$ through only three random variables α_1, α_2, and $|\tilde{\mathbf{W}}|$.

Lemma 2.1 presents a stochastic representation, which is used to characterize the joint distribution of α_1, α_2, and $|\tilde{\mathbf{W}}|$.

Lemma 2.1. *Let $\tilde{\mathbf{W}} \sim W_m(n, \mathbf{I}_m)$, $m \geq 2$, and $n > m-1$. For an m-dimensional deterministic vector $\hat{\mathbf{q}}$, we define $\mathbf{Q} = [\tilde{\mathbf{q}},\ \mathbf{Q}_1]$ as an $m \times m$ orthonormal matrix such that its first column is*

$\tilde{\mathbf{q}} = \hat{\mathbf{q}}/(\hat{\mathbf{q}}'\hat{\mathbf{q}})^{\frac{1}{2}}$. *Then, it holds that*

$$|\tilde{\mathbf{W}}| \stackrel{d}{=} \prod_{i=1}^{m} v_i, \tag{6}$$

$$\alpha_1 = \hat{\mathbf{q}}'\tilde{\mathbf{W}}^{-1}\tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\mathbf{W}}^{-1}\hat{\mathbf{q}} \stackrel{d}{=} \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})C_{11}}{v_1^2} + \frac{2(\hat{\mathbf{q}}'\hat{\mathbf{q}})\mathbf{C}_{21}'\mathbf{y}}{v_1^2\sqrt{v_2}} + \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})\mathbf{y}'\mathbf{C}_{22}\mathbf{y}}{v_1^2 v_2}, \tag{7}$$

$$\alpha_2 = \hat{\mathbf{q}}'\tilde{\mathbf{W}}^{-1}\hat{\boldsymbol{\mu}} \stackrel{d}{=} \frac{1}{v_1}\left(\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}} + \frac{\sqrt{c}\mathbf{e}_1'\mathbf{y}}{\sqrt{v_2}}\right), \tag{8}$$

where $\mathbf{y} \sim N_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$, $v_i \sim \chi^2_{n-m+i}$, $i = 1, \ldots, m$, *are mutually independent,* $\mathbf{e}_1 = [1,\ \mathbf{0}_{m-2}']'$, $c = (\hat{\mathbf{q}}'\hat{\mathbf{q}})(\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}) - (\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}})^2$, *and*

$$\mathbf{C} = \mathbf{Q}'\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{Q} = \begin{bmatrix} C_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}, \quad C_{11} = \tilde{\mathbf{q}}'\tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\mathbf{q}}. \tag{9}$$

Proof of Lemma 2.1: First, we assume that $\hat{\mathbf{q}} \neq a\hat{\boldsymbol{\mu}}$ for any $a \in \mathbb{R}$. Let

$$\tilde{\mathbf{q}} = \frac{\hat{\mathbf{q}}}{(\hat{\mathbf{q}}'\hat{\mathbf{q}})^{\frac{1}{2}}}, \quad \boldsymbol{\xi} = \frac{(\mathbf{I}_m - \tilde{\mathbf{q}}\tilde{\mathbf{q}}')\hat{\boldsymbol{\mu}}}{[\hat{\boldsymbol{\mu}}'(\mathbf{I}_m - \tilde{\mathbf{q}}\tilde{\mathbf{q}}')\hat{\boldsymbol{\mu}}]^{\frac{1}{2}}} \tag{10}$$

and let $\mathbf{Q} = [\tilde{\mathbf{q}},\ \mathbf{Q}_1]$ be an $m \times m$ orthonormal matrix, where the first column of $\mathbf{Q}_1$ is equal to $\boldsymbol{\xi}$. The results of Lemma 5.2 from the appendix yield

$$|\tilde{\mathbf{W}}| \stackrel{d}{=} \prod_{i=1}^{m} v_i, \tag{11}$$

$$\tilde{\mathbf{q}}'\tilde{\mathbf{W}}^{-1}\tilde{\mathbf{q}} \stackrel{d}{=} \frac{1}{v_1}, \tag{12}$$

$$\mathbf{Q}_1'\tilde{\mathbf{W}}^{-1}\tilde{\mathbf{q}} \stackrel{d}{=} \frac{\mathbf{y}}{v_1\sqrt{v_2}}, \tag{13}$$

$$\alpha_2 \stackrel{d}{=} \frac{1}{v_1}\left(\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}} + \frac{\sqrt{c}\mathbf{e}_1'\mathbf{y}}{\sqrt{v_2}}\right), \tag{14}$$

where $\mathbf{y} \sim N_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$, $v_i \sim \chi^2_{n-m+i}$, $i = 1, \ldots, m$, are mutually independent.

It follows that

$$\alpha_1 = (\hat{\mathbf{q}}'\hat{\mathbf{q}})\tilde{\mathbf{q}}'\tilde{\mathbf{W}}^{-1}\mathbf{Q}\mathbf{Q}'\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{Q}\mathbf{Q}'\tilde{\mathbf{W}}^{-1}\tilde{\mathbf{q}} = (\hat{\mathbf{q}}'\hat{\mathbf{q}})\mathbf{h}'\mathbf{C}\mathbf{h}, \tag{15}$$

where

$$\mathbf{h} = \mathbf{Q}'\tilde{\mathbf{W}}^{-1}\tilde{\mathbf{q}} = \begin{bmatrix} \tilde{\mathbf{q}}'\tilde{\mathbf{W}}^{-1}\tilde{\mathbf{q}} \\ \mathbf{Q}_1'\tilde{\mathbf{W}}^{-1}\tilde{\mathbf{q}} \end{bmatrix} \stackrel{d}{=} \frac{1}{v_1}\begin{bmatrix} 1 \\ \frac{\mathbf{y}}{\sqrt{v_2}} \end{bmatrix}. \tag{16}$$

Hence,

$$\alpha_1 \stackrel{d}{=} \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})C_{11}}{v_1^2} + \frac{2(\hat{\mathbf{q}}'\hat{\mathbf{q}})\mathbf{C}_{21}'\mathbf{y}}{v_1^2\sqrt{v_2}} + \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})\mathbf{y}'\mathbf{C}_{22}\mathbf{y}}{v_1^2 v_2}.$$

Now suppose $\hat{\mathbf{q}} = a\hat{\boldsymbol{\mu}}$ for some $a \in \mathbb{R}$. In this case,

$$\alpha_2 = \hat{\mathbf{q}}'\tilde{\mathbf{W}}^{-1}\hat{\boldsymbol{\mu}} = \frac{1}{a}\hat{\mathbf{q}}'\tilde{\mathbf{W}}^{-1}\hat{\mathbf{q}} \stackrel{d}{=} \frac{1}{av_1}\hat{\mathbf{q}}'\hat{\mathbf{q}} = \frac{1}{v_1}\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}},$$

which is a special case of (8) corresponding to $c = 0$. $\square$

The results of Lemma 2.1 are essential in the derivation of the density of the product of a Wishart matrix and a normal vector which is summarized in Theorem 2.1.

Theorem 2.1. *Let* $\mathbf{W} \sim W_m(n, \mathbf{\Sigma}_w)$*,* $m \geq 2$*,* $n > m-1$*,* $\mathbf{z} \sim N_m(\boldsymbol{\mu}, \mathbf{\Sigma}_z)$*, and* $\mathbf{W}$ *and* $\mathbf{z}$ *are independent. Define* $\hat{\boldsymbol{\mu}} = \mathbf{\Sigma}_w^{-\frac{1}{2}}\mathbf{\Sigma}_z^{-1}\boldsymbol{\mu}$*,* $\hat{\mathbf{q}} = \mathbf{\Sigma}_w^{-\frac{1}{2}}\mathbf{q}$*,* $\tilde{\mathbf{\Sigma}} = \mathbf{\Sigma}_w^{\frac{1}{2}}\mathbf{\Sigma}_z\mathbf{\Sigma}_w^{\frac{1}{2}}$*, and* $\mathbf{Q} = [\tilde{\mathbf{q}},\ \mathbf{Q}_1]$ *as an* $m \times m$ *orthonormal matrix such that its first column is* $\tilde{\mathbf{q}} = \hat{\mathbf{q}}/(\hat{\mathbf{q}}'\hat{\mathbf{q}})^{\frac{1}{2}}$*. Then, the density of* $\mathbf{q} = \mathbf{W}\mathbf{z}$ *is given by*

$$f(\mathbf{q}) = \frac{e^{-\frac{\boldsymbol{\mu}'\mathbf{\Sigma}_z^{-1}\boldsymbol{\mu}}{2}}}{(2\pi)^{\frac{m}{2}}|\mathbf{\Sigma}_w||\mathbf{\Sigma}_z|^{\frac{1}{2}}} \frac{\Gamma(n-m+1)}{\Gamma(n-1)}$$
$$\times \int_0^\infty \int_0^\infty \frac{|\mathbf{I}_{m-1}+\mathbf{B}|^{-\frac{1}{2}} \exp\left(b_0 + \frac{\mathbf{b}_1'(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1}{2}\right)}{v_1 v_2} f_{n-m+1}(v_1) f_{n-m+2}(v_2) \mathrm{d}v_2 \mathrm{d}v_1,$$

where $f_d(\cdot)$ *denotes the density of the* χ^2*-distribution with* d *degrees of freedom,*

$$b_0 = \frac{\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}}}{v_1} - \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})C_{11}}{2v_1^2}, \tag{17}$$

$$\mathbf{b}_1 = \frac{\sqrt{c}}{v_1\sqrt{v_2}}\mathbf{e}_1 - \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})\mathbf{C}_{21}}{v_1^2\sqrt{v_2}}, \tag{18}$$

$$\mathbf{B} = \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})\mathbf{C}_{22}}{v_1^2 v_2}, \tag{19}$$

with $\mathbf{e}_1 = [1,\ \mathbf{0}_{m-2}']'$*,* $c = (\hat{\mathbf{q}}'\hat{\mathbf{q}})(\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}) - (\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}})^2$*, and* C_{11}*,* $\mathbf{C}_{21}$*, and* $\mathbf{C}_{22}$ *given in* (9)*.*

Proof of Theorem 2.1: Using the results presented in (2) and (3) together with Lemma 2.1, we obtain

$$\begin{aligned} f(\mathbf{q}) = \mathbb{E}\left(f(\mathbf{q}|\tilde{\mathbf{W}})\right) &= \frac{e^{-\frac{\boldsymbol{\mu}'\mathbf{\Sigma}_z^{-1}\boldsymbol{\mu}}{2}}}{(2\pi)^{\frac{m}{2}}|\mathbf{\Sigma}_w||\mathbf{\Sigma}_z|^{\frac{1}{2}}} \mathbb{E}\left(|\tilde{\mathbf{W}}|^{-1}\exp\left(-\frac{\alpha_1}{2}+\alpha_2\right)\right) \\ &= \frac{e^{-\frac{\boldsymbol{\mu}'\mathbf{\Sigma}_z^{-1}\boldsymbol{\mu}}{2}}}{(2\pi)^{\frac{m}{2}}|\mathbf{\Sigma}_w||\mathbf{\Sigma}_z|^{\frac{1}{2}}} \mathbb{E}\left(v_1^{-1}v_2^{-1}\exp\left(-\frac{\alpha_1}{2}+\alpha_2\right)\right)\prod_{i=3}^m \mathbb{E}\left(v_i^{-1}\right), \end{aligned}$$

where we used that α_1 and α_2 are functions of $\mathbf{y}$, v_1, and v_2, and $\mathbf{y}$, v_i, $i = 1, \ldots, m$, are mutually independent.

The application of $v_i \sim \chi^2_{n-m+i}$ yields $\mathbb{E}\left(v_i^{-1}\right) = \frac{1}{n-m+i-2}$ and, consequently,

$$\prod_{i=3}^m \mathbb{E}\left(v_i^{-1}\right) = \frac{1}{(n-m+1)\ldots(n-2)} = \frac{\Gamma(n-m+1)}{\Gamma(n-1)}.$$

The definitions of b_0, $\mathbf{b}_1$, and $\mathbf{B}$ implies that

$$-\frac{\alpha_1+\mathbf{y}'\mathbf{y}}{2} + \alpha_2 = b_0 + \mathbf{b}_1'\mathbf{y} - \frac{1}{2}\mathbf{y}'(\mathbf{I}_{m-1}+\mathbf{B})\mathbf{y}$$

and

$$\mathbb{E}\left(v_1^{-1}v_2^{-1}\exp\left(-\frac{\alpha_1}{2}+\alpha_2\right)\right) = \int_0^\infty \int_0^\infty \frac{1}{v_1v_2} I(v_1, v_2) f_{n-m+1}(v_1) f_{n-m+2}(v_2) \mathrm{d}v_2 \mathrm{d}v_1,$$

with

$$
\begin{aligned}
I(v_1,v_2) &= \int\limits_{\mathbb{R}^{m-1}} \frac{1}{(2\pi)^{(m-1)/2}} \exp\left(b_0 + \mathbf{b}_1'\mathbf{y} - \frac{1}{2}\mathbf{y}'(\mathbf{I}_{m-1}+\mathbf{B})\mathbf{y}\right) \mathrm{d}\mathbf{y} \\
&= |\mathbf{I}_{m-1}+\mathbf{B}|^{-\frac{1}{2}} \exp\left(b_0 + \frac{1}{2}\mathbf{b}_1'(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1\right) \\
&\times \int\limits_{\mathbb{R}^{m-1}} \frac{|\mathbf{I}_{m-1}+\mathbf{B}|^{\frac{1}{2}}}{(2\pi)^{(m-1)/2}} \exp\left(-\frac{1}{2}(\mathbf{y}-(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1)'(\mathbf{I}_{m-1}+\mathbf{B})(\mathbf{y}-(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1)\right) \mathrm{d}\mathbf{y} \\
&= |\mathbf{I}_{m-1}+\mathbf{B}|^{-\frac{1}{2}} \exp\left(b_0 + \frac{1}{2}\mathbf{b}_1'(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1\right),
\end{aligned}
$$

where we used that the integral in the third line is one as the area under a density function.

Hence, we obtain

$$
\begin{aligned}
f(\mathbf{q}) &= \frac{e^{-\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}_z^{-1}\boldsymbol{\mu}}{2}}}{(2\pi)^{\frac{m}{2}}|\boldsymbol{\Sigma}_w||\boldsymbol{\Sigma}_z|^{\frac{1}{2}}} \frac{\Gamma(n-m+1)}{\Gamma(n-1)} \\
&\times \int_0^\infty\int_0^\infty \frac{|\mathbf{I}_{m-1}+\mathbf{B}|^{-\frac{1}{2}} \exp\left(b_0 + \frac{\mathbf{b}_1'(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1}{2}\right)}{v_1 v_2} f_{n-m+1}(v_1) f_{n-m+2}(v_2)\mathrm{d}v_2\mathrm{d}v_1,
\end{aligned}
$$

which completes the proof of the theorem. □

The results of Theorem 2.1 show that the density $f(\mathbf{q})$ can be obtained by using a double integral when $m \geq 2$. This finding is general and it is independent of the product dimension m. To facilitate the computation of this integral, we let $\mathbf{PDP}'$ to be the spectral decomposition of $\mathbf{C}_{22}$, where $\mathbf{D} = \mathrm{Diag}(d_1, \ldots, d_{m-1})$ is a diagonal matrix of the eigenvalues of $\mathbf{C}_{22}$. We can then write

$$
|\mathbf{I}_{m-1}+\mathbf{B}| = \left|\mathbf{I}_{m-1} + \frac{\hat{\mathbf{q}}'\hat{\mathbf{q}}}{v_1^2 v_2}\mathbf{D}\right| = \prod_{i=1}^{m-1}\left(1 + \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})d_i}{v_1^2 v_2}\right), \tag{20}
$$

$$
\begin{aligned}
\mathbf{b}_1'(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1 &= \mathbf{b}_1'\mathbf{P}\left(\mathbf{I}_{m-1} + \frac{\hat{\mathbf{q}}'\hat{\mathbf{q}}}{v_1^2 v_2}\mathbf{D}\right)^{-1}\mathbf{P}'\mathbf{b}_1 \\
&= \sum_{i=1}^{m-1} \frac{\tilde{b}_{1,i}^2}{1 + \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})d_i}{v_1^2 v_2}} = \sum_{i=1}^{m-1} \frac{\tilde{b}_{1,i}^2 v_1^2 v_2}{v_1^2 v_2 + (\hat{\mathbf{q}}'\hat{\mathbf{q}})d_i},
\end{aligned} \tag{21}
$$

where $\tilde{\mathbf{b}}_1 = \mathbf{P}'\mathbf{b}_1 = (\tilde{b}_{1,1}, \ldots, \tilde{b}_{1,m-1})'$.

3 Linear combinations

In this section, we derive the density function of p linear combinations of the product of a Wishart matrix and a normal vector expressed as $\mathbf{q} = \mathbf{L}'\mathbf{W}\mathbf{z}$, where $\mathbf{L}$ is a $m \times p$ matrix of rank p. If $p = m$, then the application of Theorem 2.1 yields the density of $\mathbf{q}$ by noting that

$$
\mathbf{q} = \mathbf{L}'\mathbf{W}\mathbf{L}\mathbf{L}^{-1}\mathbf{z} = \mathbf{W}^*\mathbf{z}^*,
$$

where $\mathbf{W}^* \sim W_m(n, \boldsymbol{\Sigma}_w^*)$ and $\mathbf{z}^* \sim N_m(\boldsymbol{\mu}^*, \boldsymbol{\Sigma}_z^*)$ with $\boldsymbol{\Sigma}_w^* = \mathbf{L}'\boldsymbol{\Sigma}_w\mathbf{L}$, $\boldsymbol{\mu}^* = \mathbf{L}^{-1}\boldsymbol{\mu}$, and $\boldsymbol{\Sigma}_z^* = \mathbf{L}^{-1}\boldsymbol{\Sigma}_z(\mathbf{L}')^{-1}$. Hence, the density function of $\mathbf{q}$ is obtained as

$$f(\mathbf{q}) = \frac{e^{-\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}_z^{-1}\boldsymbol{\mu}}{2}}}{(2\pi)^{\frac{m}{2}}|\boldsymbol{\Sigma}_w^*||\boldsymbol{\Sigma}_z^*|^{\frac{1}{2}}} \frac{\Gamma(n-m+1)}{\Gamma(n-1)}$$
$$\times \int_0^\infty\int_0^\infty \frac{|\mathbf{I}_{m-1}+\mathbf{B}^*|^{-\frac{1}{2}} \exp\left(b_0^* + \frac{(\mathbf{b}_1^*)'(\mathbf{I}_{m-1}+\mathbf{B}^*)^{-1}\mathbf{b}_1^*}{2}\right)}{v_1 v_2} f_{n-m+1}(v_1) f_{n-m+2}(v_2) dv_2 dv_1, \tag{22}$$

where b_0^*, $\mathbf{b}_1^*$, and $\mathbf{B}^*$ are obtained as b_0, $\mathbf{b}_1$, and $\mathbf{B}$ in (17) to (19) by replacing $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}_z$, and $\boldsymbol{\Sigma}_w$ with $\boldsymbol{\mu}^*$, $\boldsymbol{\Sigma}_z^*$, and $\boldsymbol{\Sigma}_w^*$, respectively.

If $p < m$, then the results of Theorem 2.1 cannot be applied directly. The derivation of the results is based on the following lemma

Lemma 3.1. *Let $\tilde{\mathbf{W}} \sim W_m(n, \boldsymbol{I}_m)$, $m \geq 2$, $n > m-1$, and let $\tilde{\mathbf{z}} \sim N_m(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}_z)$. Define $\breve{\mathbf{z}} = \tilde{\mathbf{z}}/(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{\frac{1}{2}}$ and let $\mathbf{Q}$ be an $m \times (m-1)$ orthonormal matrix with its columns orthogonal to $\tilde{\mathbf{z}}$. Then, it holds that*

$$\breve{\mathbf{z}}'\tilde{\boldsymbol{W}}\breve{\mathbf{z}} \stackrel{d}{=} u_1, \tag{23}$$
$$\mathbf{Q}'\tilde{\mathbf{W}}\breve{\mathbf{z}} \stackrel{d}{=} \sqrt{u_1}\mathbf{x}, \tag{24}$$

where $\mathbf{x} \sim N(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$, $u_1 \sim \chi^2_n$, and $\mathbf{x}$, u_1, and $\tilde{z}$ are mutually independent.

Proof of Lemma 3.1: Since $[\breve{\mathbf{z}}\ \mathbf{Q}]$ is an orthonormal matrix, the application of Theorem 3.2.5 in Muirhead [1982] yields that conditionally on $\tilde{\mathbf{z}}$

$$[\breve{\mathbf{z}}\ \mathbf{Q}]'\tilde{\mathbf{W}}[\breve{\mathbf{z}}\ \mathbf{Q}] = \begin{bmatrix} \breve{\mathbf{z}}'\tilde{\mathbf{W}}\breve{\mathbf{z}} & \breve{\mathbf{z}}'\tilde{\mathbf{W}}\mathbf{Q} \\ \mathbf{Q}'\tilde{\mathbf{W}}\breve{\mathbf{z}} & \mathbf{Q}'\tilde{\mathbf{W}}\mathbf{Q} \end{bmatrix}$$

is $W_m(n, \mathbf{I}_m)$, which does not depend on $\tilde{\mathbf{z}}$. Hence, $[\breve{\mathbf{z}}\ \mathbf{Q}]'\tilde{\mathbf{W}}[\breve{\mathbf{z}}\ \mathbf{Q}]$ is $W_m(n, \mathbf{I}_m)$ unconditionally and it is independent of $\tilde{\mathbf{z}}$.

Furthermore, Theorem 3.2.10 in Muirhead [1982] leads to

$$\breve{\mathbf{z}}'\tilde{\mathbf{W}}\breve{\mathbf{z}} \sim \chi^2_n$$

and $\mathbf{Q}'\tilde{\mathbf{W}}\breve{\mathbf{z}}|\breve{\mathbf{z}}'\tilde{\mathbf{W}}\breve{\mathbf{z}} = u_1 \sim N_{m-1}(\mathbf{0}_{m-1}, u_1\boldsymbol{I}_{m-1})$, finishing the proof of the lemma. □

We now consider the general case of $\mathbf{q} = \mathbf{L}'\mathbf{W}\mathbf{z}$, where $\mathbf{L}$ is an $m \times p$ matrix, with rank $p < m$. Define $\tilde{\mathbf{L}} = \boldsymbol{\Sigma}_w^{\frac{1}{2}}\mathbf{L}$, $\tilde{\mathbf{W}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\mathbf{W}\boldsymbol{\Sigma}_w^{-\frac{1}{2}} \sim W_m(n, \mathbf{I}_m)$ and $\tilde{\mathbf{z}} = \boldsymbol{\Sigma}_w^{\frac{1}{2}}\mathbf{z} \sim N_m(\boldsymbol{\Sigma}_w^{\frac{1}{2}}\boldsymbol{\mu}, \boldsymbol{\Sigma}_w^{\frac{1}{2}}\boldsymbol{\Sigma}_z\boldsymbol{\Sigma}_w^{\frac{1}{2}})$. Let $\mathbf{Q}$ be an $m \times (m-1)$ orthonormal matrix with its columns orthogonal to $\tilde{\mathbf{z}}$. Then, the application of Lemma 3.1 yields

$$\begin{aligned} \mathbf{q} &= \mathbf{L}'\mathbf{W}\mathbf{z} = \tilde{\mathbf{L}}'\tilde{\mathbf{W}}\tilde{\mathbf{z}} = [\tilde{\mathbf{L}}'\tilde{\mathbf{z}},\ (\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{\frac{1}{2}}\tilde{\mathbf{L}}'\mathbf{Q}] \begin{bmatrix} \frac{\tilde{\mathbf{z}}'\tilde{\mathbf{W}}\tilde{\mathbf{z}}}{\tilde{\mathbf{z}}'\tilde{\mathbf{z}}} \\ \frac{\mathbf{Q}'\tilde{\mathbf{W}}\tilde{\mathbf{z}}}{(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{\frac{1}{2}}} \end{bmatrix} \\ &\stackrel{d}{=} u_1\tilde{\mathbf{L}}'\tilde{\mathbf{z}} + (\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{\frac{1}{2}}\sqrt{u_1}\tilde{\mathbf{L}}'\mathbf{Q}\mathbf{x} \stackrel{d}{=} u_1\tilde{\mathbf{L}}'\tilde{\mathbf{z}} + (\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{\frac{1}{2}}\sqrt{u_1}\mathbf{A}^{\frac{1}{2}}\mathbf{x}_0, \end{aligned} \tag{25}$$

where $\mathbf{x}_0 \sim N(\mathbf{0}_p, \mathbf{I}_p)$, $u_1 \sim \chi^2_n$, and $\tilde{\mathbf{z}}$ are mutually independent, and

$$\mathbf{A} = \tilde{\mathbf{L}}'\mathbf{Q}\mathbf{Q}'\tilde{\mathbf{L}} = (\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{\frac{1}{2}}\left(\mathbf{I}_p - \frac{\hat{\mathbf{z}}\hat{\mathbf{z}}'}{\hat{\mathbf{z}}'\hat{\mathbf{z}} + \hat{u}}\right)(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{\frac{1}{2}}, \tag{26}$$

with

$$\hat{\mathbf{z}} = (\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-\frac{1}{2}}\tilde{\mathbf{L}}'\tilde{\mathbf{z}}, \tag{27}$$

$$\hat{u} = \tilde{\mathbf{z}}'(\mathbf{I}_m - \tilde{\mathbf{L}}(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-1}\tilde{\mathbf{L}}')\tilde{\mathbf{z}} = \tilde{\mathbf{z}}'\tilde{\mathbf{z}} - \hat{\mathbf{z}}'\hat{\mathbf{z}}, \tag{28}$$

where it is used that $\mathbf{QQ}' = \mathbf{I}_m - \tilde{\mathbf{z}}(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{-1}\tilde{\mathbf{z}}'$.

The application of the stochastic representation (25) yields the density function of $\mathbf{q}$, conditional on $\hat{\mathbf{z}}$, $\hat{u}$ and u_1, expressed as

$$f(\mathbf{q}|\hat{\mathbf{z}}, \hat{u}, u_1) = \frac{1}{(2\pi)^{\frac{p}{2}}|\mathbf{\Omega}|^{\frac{1}{2}}} \exp\left(-\frac{(\mathbf{q}-\boldsymbol{\eta})'\mathbf{\Omega}^{-1}(\mathbf{q}-\boldsymbol{\eta})}{2}\right), \tag{29}$$

where $\boldsymbol{\eta} = u_1(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{\frac{1}{2}}\hat{\mathbf{z}}$ and $\mathbf{\Omega} = u_1(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})\mathbf{A}$, which depends on $\hat{\mathbf{z}}$, $\hat{u}$, and u_1 through only four variables: $\mathbf{q}'\mathbf{\Omega}^{-1}\mathbf{q}$, $\mathbf{q}'\mathbf{\Omega}^{-1}\boldsymbol{\eta}$, $\boldsymbol{\eta}'\mathbf{\Omega}^{-1}\boldsymbol{\eta}$, and $|\mathbf{\Omega}|$.

From the definition of $\mathbf{\Omega}$ and (28), we get

$$\mathbf{\Omega}^{-1} = \frac{1}{u_1(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})}\mathbf{A}^{-1} = \frac{1}{u_1(\hat{\mathbf{z}}'\hat{\mathbf{z}}+\hat{u})}(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-\frac{1}{2}}\left(\mathbf{I}_p + \frac{\hat{\mathbf{z}}\hat{\mathbf{z}}'}{\hat{u}}\right)(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-\frac{1}{2}}, \tag{30}$$

$$|\mathbf{\Omega}| = u_1^p(\hat{\mathbf{z}}'\hat{\mathbf{z}}+\hat{u})^p\frac{|\tilde{\mathbf{L}}'\tilde{\mathbf{L}}|\hat{u}}{\hat{u}+\hat{\mathbf{z}}'\hat{\mathbf{z}}} = u_1^p(\hat{\mathbf{z}}'\hat{\mathbf{z}}+\hat{u})^{p-1}|\tilde{\mathbf{L}}'\tilde{\mathbf{L}}|\hat{u}. \tag{31}$$

Let

$$\mathbf{PDP}' = \mathbf{\Sigma}_z^{\frac{1}{2}}\mathbf{\Sigma}_w^{\frac{1}{2}}(\mathbf{I}_m - \tilde{\mathbf{L}}(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-1}\tilde{\mathbf{L}}')\mathbf{\Sigma}_w^{\frac{1}{2}}\mathbf{\Sigma}_z^{\frac{1}{2}}, \tag{32}$$

where $\mathbf{D} = \mathrm{Diag}(d_1, \ldots, d_m)$ are the eigenvalues in descending order (with $d_{m-p+1} = \ldots = d_m = 0$) and the columns of $\mathbf{P}$ are the corresponding eigenvectors. Define $\hat{\mathbf{q}} = (\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-\frac{1}{2}}\mathbf{q}$, $\hat{\mathbf{L}} = \mathbf{P}'\mathbf{\Sigma}_z^{\frac{1}{2}}\mathbf{\Sigma}_w^{\frac{1}{2}}\tilde{\mathbf{L}}(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-\frac{1}{2}}$, and

$$\mathbf{y} = \mathbf{P}'\mathbf{\Sigma}_z^{-\frac{1}{2}}\mathbf{z} \sim N(\boldsymbol{\mu}_y, \mathbf{I}_m), \quad \boldsymbol{\mu}_y = \mathbf{P}'\mathbf{\Sigma}_z^{-\frac{1}{2}}\boldsymbol{\mu}. \tag{33}$$

Then, it holds that

$$\begin{aligned}\hat{u} &= \tilde{\mathbf{z}}'(\mathbf{I}_m - \tilde{\mathbf{L}}(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-1}\tilde{\mathbf{L}}')\tilde{\mathbf{z}} = \mathbf{z}'\mathbf{\Sigma}_z^{-\frac{1}{2}}\mathbf{\Sigma}_z^{\frac{1}{2}}\mathbf{\Sigma}_w^{\frac{1}{2}}(\mathbf{I}_m - \tilde{\mathbf{L}}(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-1}\tilde{\mathbf{L}}')\mathbf{\Sigma}_w^{\frac{1}{2}}\mathbf{\Sigma}_z^{\frac{1}{2}}\mathbf{\Sigma}_z^{-\frac{1}{2}}\mathbf{z}\\ &= \mathbf{z}'\mathbf{\Sigma}_z^{-\frac{1}{2}}\mathbf{PDP}'\mathbf{\Sigma}_z^{-\frac{1}{2}}\mathbf{z} = \mathbf{y}'\mathbf{D}\mathbf{y} = \sum_{i=1}^{m-p} d_i y_i^2,\end{aligned} \tag{34}$$

$$\begin{aligned}\hat{\mathbf{z}} &= (\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-\frac{1}{2}}\tilde{\mathbf{L}}'\tilde{\mathbf{z}} = (\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-\frac{1}{2}}\tilde{\mathbf{L}}'\mathbf{\Sigma}_w^{\frac{1}{2}}\mathbf{\Sigma}_z^{\frac{1}{2}}\mathbf{PP}'\mathbf{\Sigma}_z^{-\frac{1}{2}}\mathbf{\Sigma}_w^{-\frac{1}{2}}\tilde{\mathbf{z}}\\ &= (\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-\frac{1}{2}}\tilde{\mathbf{L}}'\mathbf{\Sigma}_w^{\frac{1}{2}}\mathbf{\Sigma}_z^{\frac{1}{2}}\mathbf{P}\mathbf{y} = \hat{\mathbf{L}}'\mathbf{y}.\end{aligned} \tag{35}$$

Hence,

$$\theta_1 \equiv \mathbf{q}'\mathbf{\Omega}^{-1}\mathbf{q} = \frac{1}{u_1(\hat{\mathbf{z}}'\hat{\mathbf{z}}+\hat{u})}\left(\hat{\mathbf{q}}'\hat{\mathbf{q}} + \frac{(\hat{\mathbf{q}}'\hat{\mathbf{z}})^2}{\hat{u}}\right), \tag{36}$$

$$\theta_2 \equiv \mathbf{q}'\mathbf{\Omega}^{-1}\boldsymbol{\eta} = \frac{1}{u_1(\hat{\mathbf{z}}'\hat{\mathbf{z}}+\hat{u})}\left(u_1\hat{\mathbf{q}}'\hat{\mathbf{z}} + \frac{u_1\hat{\mathbf{q}}'\hat{\mathbf{z}}(\hat{\mathbf{z}}'\hat{\mathbf{z}})}{\hat{u}}\right) = \frac{\hat{\mathbf{q}}'\hat{\mathbf{z}}}{\hat{u}}, \tag{37}$$

$$\theta_3 \equiv \boldsymbol{\eta}'\mathbf{\Omega}^{-1}\boldsymbol{\eta} = \frac{1}{u_1(\hat{\mathbf{z}}'\hat{\mathbf{z}}+\hat{u})}u_1^2\hat{\mathbf{z}}'\left(\mathbf{I}_p + \frac{\hat{\mathbf{z}}\hat{\mathbf{z}}'}{\hat{u}}\right)\hat{\mathbf{z}} = \frac{u_1(\hat{\mathbf{z}}'\hat{\mathbf{z}})}{\hat{u}}. \tag{38}$$

Note that θ_1, θ_2, θ_3, and $|\boldsymbol{\Omega}|$ are only functions of $v_1 \equiv \hat{\mathbf{q}}'\hat{\mathbf{z}}$, $v_2 \equiv \hat{\mathbf{z}}'\hat{\mathbf{z}}$, $\hat{u}$, and u_1. Thus,

$$f(\mathbf{q}|\hat{z}, \hat{u}, u_1) = f(q|v_1, v_2, \hat{u}, u_1) = \frac{1}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Omega}|^{\frac{1}{2}}} \exp\left(-\frac{\theta_1 - 2\theta_2 + \theta_3}{2}\right), \tag{39}$$

and, consequently,

$$f_q(\mathbf{q}) = \int_0^\infty \int_0^\infty \int_0^\infty \int_{-\infty}^\infty f(\mathbf{q}|v_1, v_2, \hat{u}, u_1) f_{v_1,v_2,\hat{u}}(v_1, v_2, \hat{u}) f_n(u_1) \mathrm{d}v_1 \mathrm{d}v_2 \mathrm{d}\hat{u} \mathrm{d}u_1. \tag{40}$$

To reduce the above expression to a triple integral, we make use of the following identity

$$\int_0^\infty u_1^{-\frac{p}{2}} \exp\left(-\frac{a}{2u_1} - \frac{bu_1}{2}\right) f_n(u_1)\mathrm{d}u_1 = \frac{1}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)} \left(\frac{a}{1+b}\right)^{\frac{n-p}{4}} K_{\frac{n-p}{2}}\left(\sqrt{a(1+b)}\right) \tag{41}$$

for $a \neq 0$, where $K_\nu(x)$ is the modified Bessel function of the second kind. When $a = 0$, we have

$$\int_0^\infty u_1^{-\frac{p}{2}} \exp\left(-\frac{bu_1}{2}\right) f_n(u_1)\mathrm{d}u_1 = \frac{\Gamma\left(\frac{n-p}{2}\right)}{2^{\frac{p}{2}}\Gamma\left(\frac{n}{2}\right)(1+b)^{\frac{n-p}{2}}}. \tag{42}$$

Since u_1 is independent of $\tilde{\mathbf{z}}$ and, consequently, of v_1, v_2, and $\hat{u}$, the identity (41) yields

$$\begin{aligned} f(\mathbf{q}|v_1, v_2, \hat{u}) &= \int_0^\infty f(\mathbf{q}|v_1, v_2, \hat{u}, u_1) f_n(u_1)\mathrm{d}u_1 \\ &= \frac{\exp\left(\frac{\hat{\mathbf{q}}'\hat{\mathbf{z}}}{\hat{u}}\right)\left[(\hat{\mathbf{q}}'\hat{\mathbf{q}})\hat{u} + (\hat{\mathbf{q}}'\hat{\mathbf{z}})^2\right]^{\frac{n-p}{4}}}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)(2\pi)^{\frac{p}{2}}|\tilde{\mathbf{L}}'\tilde{\mathbf{L}}|^{\frac{1}{2}}\hat{u}^{\frac{1}{2}}(\hat{\mathbf{z}}'\hat{\mathbf{z}} + \hat{u})^{\frac{n-1}{2}}} K_{\frac{n-p}{2}}\left(\frac{\left[(\hat{\mathbf{q}}'\hat{\mathbf{q}})\hat{u} + (\hat{\mathbf{q}}'\hat{\mathbf{z}})^2\right]^{\frac{1}{2}}}{\hat{u}}\right) \end{aligned} \tag{43}$$

when $\mathbf{q} \neq \mathbf{0}_p$. As a result, we have

$$f_q(\mathbf{q}) = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty f(\mathbf{q}|v_1, v_2, \hat{u}) f_{v_1,v_2,\hat{u}}(v_1, v_2, \hat{u})\mathrm{d}v_1\mathrm{d}v_2\mathrm{d}\hat{u} \quad \text{for } \mathbf{q} \neq \mathbf{0}_p. \tag{44}$$

When $\mathbf{q} = \mathbf{0}_p$, we have $v_1 = 0$ and

$$f(\mathbf{0}_p|v_2, \hat{u}) = \int_0^\infty f(q|v_2, \hat{u}, u_1) f_n(u_1)\mathrm{d}u_1 = \frac{\Gamma\left(\frac{n-p}{2}\right)\hat{u}^{\frac{n-p-1}{2}}}{2^p\pi^{\frac{p}{2}}\Gamma\left(\frac{n}{2}\right)|\tilde{\mathbf{L}}'\tilde{\mathbf{L}}|^{\frac{1}{2}}(v_2 + \hat{u})^{\frac{n-1}{2}}}. \tag{45}$$

As a result, we have

$$f_q(\mathbf{0}_p) = \int_0^\infty \int_0^\infty f(\mathbf{0}_p|v_2, \hat{u}) f_{v_2,\hat{u}}(v_2, \hat{u})\mathrm{d}v_2\mathrm{d}\hat{u} = \frac{\Gamma\left(\frac{n-p}{2}\right)}{2^p\pi^{\frac{p}{2}}\Gamma\left(\frac{n}{2}\right)|\tilde{\mathbf{L}}'\tilde{\mathbf{L}}|^{\frac{1}{2}}} E\left[\frac{\hat{u}^{\frac{n-p-1}{2}}}{(v_2 + \hat{u})^{\frac{n-1}{2}}}\right]. \tag{46}$$

This is the expectation of a ratio of two quadratic forms in multivariate normal random variables, raised to different powers. Using the method in Bao and Kan [2013], this expectation can be computed using a single or a double integral, depending on whether $(n-p-1)/2$ is an integer or not.

Finally, we note that the evaluation of the density $f_{v_1,v_2,\hat{u}}(v_1, v_2, \hat{u})$ is related to the derivation of joint density function of linear and quadratic forms in normally distributed random variables (see, e.g., Mathai and Provost [1992], Schöne and Schmid [2000]). Next, we present the results in several important special cases: $p = 1$ in Section 3.1 and $p = 2$ in Section 3.2.

3.1 Special case: One linear combination

When $p = 1$ and $q = \boldsymbol{\alpha}'\mathbf{W}\mathbf{z}$, we have $v_1 = \hat{q}\hat{z}$ and $v_2 = \hat{z}^2$. By writing $\mathbf{L} = \boldsymbol{\alpha}$, the triple integral (44) is reduced to the following double integral for $q \neq 0$

$$f_q(q) = \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{|\hat{q}|^{\frac{n-1}{2}} \exp\left(\frac{\hat{q}\hat{z}}{\hat{u}}\right)}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)\sqrt{\pi}(\alpha'\Sigma_w\alpha)^{\frac{1}{2}}\hat{u}^{\frac{1}{2}}(\hat{z}^2+\hat{u})^{\frac{n-1}{4}}} K_{\frac{n-1}{2}}\left(\frac{|\hat{q}|(\hat{z}^2+\hat{u})^{\frac{1}{2}}}{\hat{u}}\right) f_{\hat{u},\hat{z}}(\hat{u},\hat{z})\mathrm{d}\hat{u}\mathrm{d}\hat{z}. \tag{47}$$

When $q = 0$, we can use (46) to obtain

$$f_q(0) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{n}{2}\right)\sqrt{\pi}(\alpha'\Sigma_w\alpha)^{\frac{1}{2}}} E\left[\frac{\hat{u}^{\frac{n-2}{2}}}{(\hat{z}^2+\hat{u})^{\frac{n-1}{2}}}\right]. \tag{48}$$

To evaluate the density of q using the above expression, we need the joint density of $(\hat{u}, \hat{z})$. Note that

$$\hat{z} \sim N\left(\frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w\boldsymbol{\mu}}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w\boldsymbol{\alpha})^{\frac{1}{2}}}, \frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w\boldsymbol{\Sigma}_z\boldsymbol{\Sigma}_w\boldsymbol{\alpha}}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w\boldsymbol{\alpha}}\right). \tag{49}$$

Next, we show that conditional on $\hat{z}$, $\hat{u}$ is a quadratic form in multivariate normal random variables.

Let $\hat{\boldsymbol{\alpha}} = \mathbf{P}'\boldsymbol{\Sigma}_z^{\frac{1}{2}}\boldsymbol{\Sigma}_w\boldsymbol{\alpha}$ and let $\mathbf{Q} = [\hat{\boldsymbol{\alpha}}/(\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\alpha}})^{\frac{1}{2}}, \mathbf{Q}_1]$ be an $m \times m$ orthonormal matrix. Define $\mathbf{x} = \mathbf{Q}'\mathbf{y} \sim N_m(\mathbf{Q}'\boldsymbol{\mu}_y, \mathbf{I}_m)$. Then, it holds that

$$\hat{z} = \frac{\hat{\boldsymbol{\alpha}}'\mathbf{y}}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w\boldsymbol{\alpha})^{\frac{1}{2}}} = \left(\frac{\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w\boldsymbol{\alpha}}\right)^{\frac{1}{2}} x_1 = \left(\frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w\boldsymbol{\Sigma}_z\boldsymbol{\Sigma}_w\boldsymbol{\alpha}}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w\boldsymbol{\alpha}}\right)^{\frac{1}{2}} x_1, \tag{50}$$

$$\hat{u} = \mathbf{y}'\mathbf{D}\mathbf{y} = \mathbf{y}'\mathbf{Q}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{Q}'\mathbf{y} = \mathbf{x}'\mathbf{C}\mathbf{x}, \tag{51}$$

where

$$\mathbf{Q}'\mathbf{D}\mathbf{Q} = \mathbf{C} = \begin{bmatrix} C_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}, \tag{52}$$

with C_{11} the (1,1) element of $\mathbf{C}$. Denoting $\tilde{\mathbf{x}} = [x_2, \ldots, x_m]'$, we can then write

$$\hat{u} = C_{11}x_1^2 + 2x_1\mathbf{C}_{21}'\tilde{\mathbf{x}} + \tilde{\mathbf{x}}'\mathbf{C}_{22}\tilde{\mathbf{x}} = C_{11}\frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w\boldsymbol{\alpha}}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w\boldsymbol{\Sigma}_z\boldsymbol{\Sigma}_w\boldsymbol{\alpha}}\hat{z}^2 + 2\left(\frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w\boldsymbol{\alpha}}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w\boldsymbol{\Sigma}_z\boldsymbol{\Sigma}_w\boldsymbol{\alpha}}\right)^{\frac{1}{2}}\hat{z}\mathbf{C}_{21}'\tilde{\mathbf{x}} + \tilde{\mathbf{x}}'\mathbf{C}_{22}\tilde{\mathbf{x}}. \tag{53}$$

Hence, conditional on $\hat{z}$, $\hat{u}$ is a quadratic form in $\tilde{\mathbf{x}}$, and the evaluation of its density is discussed in the appendix (see Section 5.2). Finally, the density of q is evaluated as

$$f_q(q) = \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{|\hat{q}|^{\frac{n-1}{2}} \exp\left(\frac{\hat{q}\hat{z}}{\hat{u}}\right) K_{\frac{n-1}{2}}\left(\frac{|\hat{q}|(\hat{z}^2+\hat{u})^{\frac{1}{2}}}{\hat{u}}\right)}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)\sqrt{\pi}(\alpha'\Sigma_w\alpha)^{\frac{1}{2}}\hat{u}^{\frac{1}{2}}(\hat{z}^2+\hat{u})^{\frac{n-1}{4}}} f_{\hat{u}|\hat{z}}(\hat{u})\mathrm{d}\hat{u} f_{\hat{z}}(\hat{z})\mathrm{d}\hat{z}, \quad q \neq 0. \tag{54}$$

In Figure 1, we plot the density function of $q = \boldsymbol{\alpha}'\mathbf{W}\mathbf{z}$ when $m = 5$ and $n = \{9, 10\}$. We set $\boldsymbol{\alpha} = \mathbf{1}_5$, $\boldsymbol{\mu} = [1,\ 2,\ 3,\ 4,\ 5]'$, $\boldsymbol{\Sigma}_w = 0.5\mathbf{I}_5 + 0.5\mathbf{1}_5\mathbf{1}_5'$, and $\boldsymbol{\Sigma}_z = 0.8\mathbf{I}_5 + 0.2\mathbf{1}_5\mathbf{1}_5'$. The density of q corresponding to $n = 9$ is plotted as a solid line, while the dotted line corresponds to the case of $n = 10$. Both density functions are skewed to the right and they are unimodal. When n is larger, the density function shifts to the right and it possesses a lower peak.

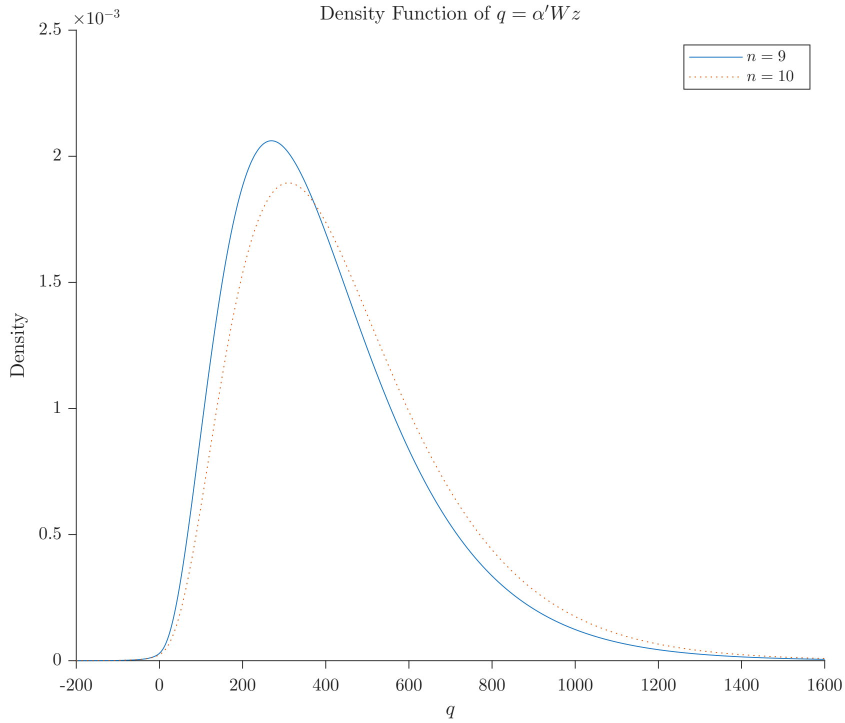

Figure 1: Density function of $q = \boldsymbol{\alpha}'\mathbf{Wz}$ for the cases of $n = 9$ and $n = 10$ when $\boldsymbol{\alpha} = \mathbf{1}_5$, $\boldsymbol{\mu} = [1,\ 2,\ 3,\ 4,\ 5]'$, $\boldsymbol{\Sigma}_w = 0.5\mathbf{I}_5 + 0.5\mathbf{1}_5\mathbf{1}_5'$, and $\boldsymbol{\Sigma}_z = 0.8\mathbf{I}_5 + 0.2\mathbf{1}_5\mathbf{1}_5'$.

3.2 Special case: Two linear combinations

When $p = 2$, we have $v_1 = \hat{q}_1\hat{z}_1 + \hat{q}_2\hat{z}_2$ and $v_2 = \hat{z}_1^2 + \hat{z}_2^2$, where $\hat{\mathbf{z}} = \hat{\mathbf{L}}'\mathbf{y}$. Hence,

$$f_q(\mathbf{q}) = \mathbb{E}[f_q(\mathbf{q}|v_1, v_2, \hat{u})] = \mathbb{E}[f_q(\mathbf{q}|\hat{z}_1, \hat{z}_2, \hat{u})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\exp\left(\frac{v_1}{\hat{u}}\right)[(\hat{\mathbf{q}}'\hat{\mathbf{q}})\hat{u} + v_1^2]^{\frac{n-2}{4}}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\pi|\tilde{\mathbf{L}}'\tilde{\mathbf{L}}|^{\frac{1}{2}}\hat{u}^{\frac{1}{2}}(v_2 + \hat{u})^{\frac{n-1}{2}}}$$
$$\times K_{\frac{n-2}{2}}\left(\frac{[(\hat{\mathbf{q}}'\hat{\mathbf{q}})\hat{u} + v_1^2]^{\frac{1}{2}}}{\hat{u}}\right) f(\hat{u}, \hat{z}_1, \hat{z}_2)\mathrm{d}\hat{z}_1\mathrm{d}\hat{z}_2\mathrm{d}\hat{u}. \tag{55}$$

Let $\mathbf{Q} = [\hat{\mathbf{L}}(\hat{\mathbf{L}}'\hat{\mathbf{L}})^{-\frac{1}{2}},\ \mathbf{Q}_1]$ be an $m \times m$ orthonormal matrix, where the columns of $\mathbf{Q}_1$ are orthogonal to $\hat{\mathbf{L}}$. Define $\mathbf{x} = \mathbf{Q}'\mathbf{y} \sim N_m(\mathbf{Q}'\boldsymbol{\mu}_y, \mathbf{I}_m)$. We can then write

$$\hat{\mathbf{z}} = \hat{\mathbf{L}}'\mathbf{y} = \hat{\mathbf{L}}'\mathbf{Q}\mathbf{Q}'\mathbf{y} = \hat{\mathbf{L}}'\mathbf{Q}\mathbf{x} = (\hat{\mathbf{L}}'\hat{\mathbf{L}})^{\frac{1}{2}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{56}$$

$$\hat{u} = \mathbf{y}'\mathbf{D}\mathbf{y} = \mathbf{x}'\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{x} = \mathbf{x}'\mathbf{C}\mathbf{x}, \tag{57}$$

with

$$\mathbf{Q}'\mathbf{D}\mathbf{Q} \equiv \mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}, \tag{58}$$

where $\mathbf{C}_{11}$ is 2×2. Conditional on $(\hat{z}_1, \hat{z}_2)$, or equivalently on (x_1, x_2), we can write $\hat{u}$ as a quadratic form in $\tilde{\mathbf{x}} = [x_3, \ldots, x_m]'$ as

$$\hat{u} = [x_1,\ x_2]\mathbf{C}_{11}[x_1,\ x_2]' + 2[x_1,\ x_2]\mathbf{C}_{21}'\tilde{\mathbf{x}} + \tilde{\mathbf{x}}'\mathbf{C}_{22}\tilde{\mathbf{x}}. \tag{59}$$

Since the relationship between $(\hat{z}_1, \hat{z}_2)$ and (x_1, x_2) is one-to-one, we get the density of $\mathbf{q}$ as in (55) conditionally on (x_1, x_2) where $f(\hat{u}, x_1, x_2) = f(\hat{u}|x_1, x_2)f(x_1)f(x_2)$, and the evaluation of $f(\hat{u}|x_1, x_2)$ is discussed in the appendix (see Section 5.2).

3.3 Special case: Identity covariance matrices

In this subsection, we assume that $\mathbf{\Sigma}_w = \mathbf{\Sigma}_z = \mathbf{I}_m$. Then, we get that $\hat{\mathbf{z}}$ and $\hat{u}$ are independent with (see Theorem 5.1.3 and Theorem 5.5.1 in Mathai and Provost [1992])

$$\hat{\mathbf{z}} = (\mathbf{L}'\mathbf{L})^{-\frac{1}{2}}\mathbf{L}'\mathbf{z} \sim N_p(\hat{\boldsymbol{\mu}}, \mathbf{I}_p), \quad \hat{\boldsymbol{\mu}} = (\mathbf{L}'\mathbf{L})^{-\frac{1}{2}}\mathbf{L}'\boldsymbol{\mu}, \tag{60}$$

$$\hat{u} = \mathbf{z}'(\mathbf{I}_m - \mathbf{L}(\mathbf{L}'\mathbf{L})^{-1}\mathbf{L}')\mathbf{z} \sim \chi^2_{m-p}(\hat{\delta}), \quad \hat{\delta} = \boldsymbol{\mu}'\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}. \tag{61}$$

Let

$$w_1 = \frac{\hat{\mathbf{q}}'\hat{\mathbf{z}}}{\sqrt{\hat{\mathbf{q}}'\hat{\mathbf{q}}}} \quad \text{and} \quad \hat{u}_1 = \hat{\mathbf{z}}'\left(\mathbf{I}_p - \frac{\hat{\mathbf{q}}\hat{\mathbf{q}}'}{\hat{\mathbf{q}}'\hat{\mathbf{q}}}\right)\hat{\mathbf{z}}. \tag{62}$$

Then, w_1 and $\hat{u}_1$ are independent with $w_1 \sim N(\tilde{\mu}_1, 1)$, $\tilde{\mu}_1 = \frac{\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}}}{\sqrt{\hat{\mathbf{q}}'\hat{\mathbf{q}}}}$ and $\hat{u}_1 \sim \chi^2_{p-1}(\hat{\delta}_1)$, $\hat{\delta}_1 = \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}} - \tilde{\mu}_1^2$ (see Theorem 5.1.3 and Theorem 5.5.1 in Mathai and Provost [1992]). Moreover, $\hat{u}$, $\hat{u}_1$ and w_1 are mutually independent.

Then, the density of $\mathbf{q} = \mathbf{L}'\mathbf{W}\mathbf{z}$ simplifies to

$$f_q(\mathbf{q}) = \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} f(\mathbf{q}|w_1, \hat{u}_1, \hat{u})\phi(w_1 - \tilde{\mu}_1) f_{\hat{u}_1}(\hat{u}_1) f_{\hat{u}}(\hat{u}) \mathrm{d}w_1 \mathrm{d}\hat{u}_1 \mathrm{d}\hat{u} \quad \cancel{\text{for } \mathbf{q} \neq \mathbf{0}_p} \tag{63}$$

with

$$f(\mathbf{q}|w_1, \hat{u}_1, \hat{u}) = \frac{\exp\left(\frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})^{\frac{1}{2}} w_1}{\hat{u}}\right)\left[(\hat{\mathbf{q}}'\hat{\mathbf{q}})(\hat{u} + w_1^2)\right]^{\frac{n-p}{4}}}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)(2\pi)^{\frac{p}{2}}|\mathbf{L}'\mathbf{L}|^{\frac{1}{2}}\hat{u}^{\frac{1}{2}}(w_1^2 + \hat{u}_1 + \hat{u})^{\frac{n-1}{2}}} K_{\frac{n-p}{2}}\left(\frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})^{\frac{1}{2}}(\hat{u} + w_1^2)^{\frac{1}{2}}}{\hat{u}}\right) \tag{64}$$

when $\mathbf{q} \neq \mathbf{0}_p$. Finally, when $\mathbf{q} = \mathbf{0}_p$, we have

$$f_q(\mathbf{0}_p) = \frac{\Gamma\left(\frac{n-p}{2}\right)}{2^p \pi^{\frac{p}{2}} \Gamma\left(\frac{n}{2}\right) |\mathbf{L}'\mathbf{L}|^{\frac{1}{2}}} E\left[\frac{\hat{u}^{\frac{n-p-1}{2}}}{(v + \hat{u})^{\frac{n-1}{2}}}\right],$$

where $\hat{u}$ and $v = \hat{\mathbf{z}}'\hat{\mathbf{z}}$ are independent with $v \sim \chi^2_p(\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}})$. The expectation can be evaluated by using the results of Bao and Kan [2013].

When $p = 1$ and $\mathbf{L} = \boldsymbol{\alpha}$, we get that $\hat{u}_1 = 0$ and $\hat{q} = \frac{q}{\sqrt{\boldsymbol{\alpha}'\boldsymbol{\alpha}}}$. Then, the application of (60), (61), and (62) together with (40) yields

$$f_q(q) = \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} f(q|w_1, \hat{u}, u_1)\phi(w_1 - \tilde{\mu}_1) f_{\hat{u}}(\hat{u}) f_n(u_1) \mathrm{d}w_1 \mathrm{d}\hat{u} \mathrm{d}u_1 \tag{65}$$

for $q \neq 0$, where

$$f(q|w_1, \hat{u}, u_1) = \frac{1}{\sqrt{2\pi}\sqrt{u_1}(\boldsymbol{\alpha}'\boldsymbol{\alpha})^{\frac{1}{2}}\sqrt{\hat{u}}} \exp\left(-\frac{\left(\sqrt{u_1}w_1 - \frac{\hat{q}}{\sqrt{u_1}}\right)^2}{2\hat{u}}\right).$$

Using the fact that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{\left(\sqrt{u_1}w_1 - \frac{\hat{q}}{\sqrt{u_1}}\right)^2}{2\hat{u}}\right)\phi(w_1 - \tilde{\mu}_1)\mathrm{d}w_1 = \exp\left(-\frac{(\hat{q} - u_1\tilde{\mu}_1)^2}{2u_1(u_1 + \hat{u})}\right)\left(\frac{\hat{u}}{u_1 + \hat{u}}\right)^{\frac{1}{2}},$$

we can write $f(q)$ as a double integral when $p = 1$. In addition, we have $\tilde{\mu}_1 = \frac{\boldsymbol{\alpha}'\boldsymbol{\mu}}{\sqrt{\boldsymbol{\alpha}'\boldsymbol{\alpha}}}$, $\hat{q} = \frac{q}{\sqrt{\boldsymbol{\alpha}'\boldsymbol{\alpha}}}$ and, consequently,

$$(\hat{q} - u_1\tilde{\mu}_1)^2 = \frac{(q - u_1\boldsymbol{\alpha}'\boldsymbol{\mu})^2}{\boldsymbol{\alpha}'\boldsymbol{\alpha}}.$$

Hence, the density function (65) simplifies to

$$f_q(q) = \int_0^\infty \int_0^\infty \phi_{u_1\boldsymbol{\alpha}'\boldsymbol{\mu},(\boldsymbol{\alpha}'\boldsymbol{\alpha})(u_1^2+u_1\hat{u})}(q) f_{m-1}^{\hat{\delta}}(\hat{u}) f_n(u_1) \mathrm{d}\hat{u}\mathrm{d}u_1, \tag{66}$$

where ϕ_{μ,σ^2} is the density function of $N(\mu, \sigma^2)$, $f_{m-1}^{\hat{\delta}}$ is the density function of $\chi^2_{m-1}(\hat{\delta})$, and f_n is the density function of χ^2_n.

4 Summary

The Wishart and normal distributions are fundamental in multivariate statistical theory, where they arise in both parameter estimation and hypothesis testing (see Muirhead [1982]). These matrix-variate and multivariate families are well studied (see, e.g., Muirhead [1982], Gupta and Nagar [2000]) and underlie many distributional families, such as elliptically contoured distributions (Gupta et al. [2013]), skew-normal distributions (Azzalini [2013]), and the Laplace distribution (Kotz et al. [2012]). Their properties are also widely used to derive theoretical results in applications such as finance (Bodnar et al. [2022], Kan et al. [2024]) and biology (Serván et al. [2025]).

In this paper, we derived the density function of the product of a Wishart matrix and a normal vector, expressed as a two-dimensional integral. The result is obtained in the general case, without assuming any relationship between the covariance matrices of the Wishart and normal distributions, and in particular without assuming proportionality. We also derived the density of linear combinations of the product, with particular attention to the special cases of one and two linear combinations and of identity covariance matrices; the resulting expressions are given as low-dimensional integrals that are easy to evaluate in practice.

Several extensions are left for future work, including the moments and tail behavior of the proposed matrix-variate mixture family, a singular-Wishart counterpart in the spirit of Srivastava [2003], Bodnar et al. [2015], and Bodnar et al. [2018], and applications to the sampling distributions of statistics involving products of Wishart matrices and normal vectors in financial econometrics.

5 Appendix

5.1 Stochastic representation

For m-dimensional vectors $\mathbf{q}$ and $\boldsymbol{\mu}$, we derive a stochastic representation of $\mathbf{q}'\mathbf{W}^{-1}\mathbf{q}$, $\mathbf{q}'\mathbf{W}^{-1}\boldsymbol{\mu}$, and $|\mathbf{W}|$.

Lemma 5.1. *Let* $\mathbf{x} \sim N_m(\mathbf{0}_m, \mathbf{I}_m)$, $\mathbf{W} \sim W_m(n, \mathbf{I}_m)$, $m \geq 2$, $n > m - 1$, *and let* $\mathbf{x}$ *and* $\mathbf{W}$ *be independent. Define* $\mathbf{h} = \mathbf{W}^{-1/2}\mathbf{x}$, $\mathbf{P} = [\tilde{\mathbf{h}}, \mathbf{P}_1]$ *orthonormal matrix with* $\tilde{\mathbf{h}} = \mathbf{h}/\sqrt{\mathbf{h}'\mathbf{h}}$, *and*

$$\mathbf{H} = \mathbf{P}'\mathbf{W}\mathbf{P} = \begin{pmatrix} H_{11} & \mathbf{H}_{12} \\ \mathbf{H}'_{12} & \mathbf{H}_{22} \end{pmatrix}, \quad H_{11} = \tilde{\mathbf{h}}'\mathbf{W}\tilde{\mathbf{h}}.$$

Let $v_{11} = H_{11}$, $\mathbf{v}_{12} = \mathbf{H}_{12}/\sqrt{H_{11}}$, $\mathbf{V}_{22\cdot1} = \mathbf{H}_{22} - \mathbf{H}_{12}\mathbf{H}'_{12}/H_{11}$. *Then, it holds that* *(i)* $(\mathbf{h}, v_{11})$, $\mathbf{v}_{12}$, $\mathbf{V}_{22\cdot1}$ *are mutually independent;*

(ii) $\mathbf{h}|v_{11} \sim N_m(\mathbf{0}_m, \frac{1}{v_{11}}\mathbf{I}_m)$*;*

(iii) $v_{11} \sim \chi^2_{n-m+1}$*;*

(iv) $\mathbf{v}_{12} \sim N_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$*;*

(v) $\mathbf{V}_{22\cdot1} \sim W_{m-1}(n, \mathbf{I}_{m-1})$*.*

Proof of Lemma 5.1: Following the assumptions of the lemma, the joint density of $\mathbf{h} = \mathbf{W}^{-1/2}\mathbf{x}$ and $\mathbf{W}$ is given by

$$\begin{aligned} f(\mathbf{h}, \mathbf{W}) &= f(\mathbf{h}|\mathbf{W})f(\mathbf{W}) \\ &= \frac{1}{(2\pi)^{m/2}}|\mathbf{W}|^{1/2}\exp\left(-\frac{1}{2}\mathbf{h}'\mathbf{W}\mathbf{h}\right)\frac{1}{2^{mn/2}\Gamma_m\left(\frac{n}{2}\right)}|\mathbf{W}|^{(n-m-1)/2}\exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{W})\right), \end{aligned}$$

where $\Gamma_m(\cdot)$ denotes the multivariate gamma function (see, e.g., Gupta and Nagar [2000]).

Let $\mathbf{P} = [\tilde{\mathbf{h}}, \mathbf{P}_1]$ be an orthonormal matrix with $\tilde{\mathbf{h}} = \mathbf{h}/\sqrt{\mathbf{h}'\mathbf{h}}$. Making transformation $(\mathbf{h}, \mathbf{W})$ to $(\mathbf{h}, \mathbf{H})$, $\mathbf{H} = \mathbf{P}'\mathbf{W}\mathbf{P}$, with Jacobian equal to one, we get

$$f(\mathbf{h}, \mathbf{H}) = \frac{1}{(2\pi)^{m/2}}\exp\left(-\frac{1}{2}(\mathbf{P}'\mathbf{h})'\mathbf{H}(\mathbf{P}'\mathbf{h})\right)\frac{1}{2^{mn/2}\Gamma_m\left(\frac{n}{2}\right)}|\mathbf{H}|^{(n-m)/2}\exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{H})\right).$$

Using the definitions of v_{11}, $\mathbf{v}_{12}$, $\mathbf{V}_{22\cdot1}$ and the fact that the Jacobian of the transformation from $(\mathbf{h}, H_{11}, \mathbf{H}_{12}, \mathbf{H}_{22})$ to $(\mathbf{h}, v_{11}, \mathbf{v}_{12}, \mathbf{V}_{22\cdot1})$ is $v_{11}^{(m-1)/2}$, we obtain

$$\begin{aligned} f(\mathbf{h}, v_{11}, \mathbf{v}_{12}, \mathbf{V}_{22\cdot1}) &= \frac{v_{11}^{m/2}}{(2\pi)^{m/2}}\exp\left(-\frac{1}{2}v_{11}\mathbf{h}'\mathbf{h}\right) \\ &\times \frac{v_{11}^{(n-m+1)/2-1}}{2^{(n-m+1)/2}\Gamma\left(\frac{n-m+1}{2}\right)}\exp\left(-\frac{v_{11}}{2}\right) \\ &\times \frac{1}{2^{(m-1)n/2}\Gamma_{m-1}\left(\frac{n}{2}\right)}|\mathbf{V}_{22\cdot1}|^{(n-m)/2}\exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{V}_{22\cdot1})\right) \\ &\times \frac{1}{(2\pi)^{(m-1)/2}}\exp\left(-\frac{\mathbf{v}_{12}'\mathbf{v}_{12}}{2}\right), \end{aligned}$$

where the properties of the multivariate gamma function is used (see page 95 in Muirhead [1982]). The formula of $f(\mathbf{h}, v_{11}, \mathbf{v}_{12}, \mathbf{V}_{22\cdot1})$ shows that $(\mathbf{h}, v_{11})$, $\mathbf{v}_{12}$, $\mathbf{V}_{22\cdot1}$ are mutually independent and have the distributions as summarized in the statement of the lemma. □

Corollary 5.1. *Let* $\mathbf{x} \sim N_m(\mathbf{0}_m, \mathbf{I}_m)$*,* $\mathbf{W} \sim W_m(n, \mathbf{I}_m)$*,* $m \geq 2$*,* $n > m-1$*, and let* $\mathbf{x}$ *and* $\mathbf{W}$ *be independent. Define* $\mathbf{h} = \mathbf{W}^{-1/2}\mathbf{x}$*,* $\mathbf{P} = [\tilde{\mathbf{h}}, \mathbf{P}_1]$ *orthonormal matrix with* $\tilde{\mathbf{h}} = \mathbf{h}/\sqrt{\mathbf{h}'\mathbf{h}}$*, and*

$$\mathbf{H} = \mathbf{P}'\mathbf{W}\mathbf{P} = \begin{pmatrix} H_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{12}' & \mathbf{H}_{22} \end{pmatrix}, \quad H_{11} = \tilde{\mathbf{h}}'\mathbf{W}\tilde{\mathbf{h}}.$$

Let $v_{11} = H_{11}$*,* $\mathbf{v}_{12} = \mathbf{H}_{12}/\sqrt{H_{11}}$*,* $\mathbf{V}_{22\cdot1} = \mathbf{H}_{22} - \mathbf{H}_{12}\mathbf{H}_{12}'/H_{11}$*. Then, we get the following stochastic representations*

$$\mathbf{h} \stackrel{d}{=} \frac{\mathbf{y}}{\sqrt{v_1}}, \tag{67}$$

$$|\mathbf{W}| \stackrel{d}{=} \prod_{i=1}^{m} v_i, \tag{68}$$

where $\mathbf{y}$*,* v_i*,* $i = 1, \ldots, m$ *are mutually independent with* $\mathbf{y} \sim N_m(\mathbf{0}_m, \mathbf{I}_m)$ *and* $v_i \sim \chi^2_{n-m+i}$*,* $i = 1, \ldots, m$*.*

Proof of Corollary 5.1: The results of the corollary follow from Lemma 5.1, the equalities $|\mathbf{W}| = |\mathbf{H}|$ and $|\mathbf{H}| = v_{11}|\mathbf{V}_{22\cdot 1}|$, and the fact that (see Theorem 3.2.15 in Muirhead [1982])

$$|\mathbf{V}_{22\cdot 1}| \stackrel{d}{=} \prod_{i=2}^{m} v_i,$$

where v_i, $i = 2, \ldots, m$, are independent with $v_i \sim \chi^2_{n-m+i}$. □

Corollary 5.2. *Let $\mathbf{W} \sim W_m(n, \mathbf{I}_m)$, $n \geq m \geq 2$, be independent of $\mathbf{r}$, which is uniformly distributed on the unit sphere in $\mathbb{R}^m$. Then, $\frac{\mathbf{W}^{-1/2}\mathbf{r}}{\sqrt{\mathbf{r}'\mathbf{W}^{-1}\mathbf{r}}}$ is uniformly distributed on the unit sphere in $\mathbb{R}^m$, and it is independent of $\mathbf{r}'\mathbf{W}^{-1}\mathbf{r}$.*

Proof of Corollary 5.2: Let $\xi \sim \chi^2_m$ and independent of $\mathbf{W}$ and $\mathbf{r}$. Then, $\mathbf{x} = \sqrt{\xi}\mathbf{r} \sim N_m(\mathbf{0}_m, \mathbf{I}_m)$ and

$$\frac{\mathbf{W}^{-1/2}\mathbf{r}}{\sqrt{\mathbf{r}'\mathbf{W}^{-1}\mathbf{r}}} = \frac{\mathbf{W}^{-1/2}\mathbf{x}}{\sqrt{\mathbf{x}'\mathbf{W}^{-1}\mathbf{x}}}.$$

The application of Lemma 5.1 yields that $\mathbf{W}^{-1/2}\mathbf{x}$ conditionally on $v_{11} = \mathbf{h}'\mathbf{W}\mathbf{h}/(\mathbf{h}'\mathbf{h})$ is multivariate normally distributed with zero mean vector and covariance matrix proportional to the identity matrix. Since the multivariate normal distribution is elliptically contoured, then $\frac{\mathbf{W}^{-1/2}\mathbf{x}}{\sqrt{\mathbf{x}'\mathbf{W}^{-1}\mathbf{x}}}$ is uniformly distributed on the unit sphere in $\mathbb{R}^m$, and it is independent of $\mathbf{x}'\mathbf{W}^{-1}\mathbf{x} = \xi\mathbf{r}'\mathbf{W}^{-1}\mathbf{r}$ (see Theorem 2.15 in Gupta et al. [2013]). Finally, noting that ξ and $\mathbf{r}$ are independent, we get the statement of the corollary. □

Lemma 5.2. *Let $\mathbf{W} \sim W_m(n, \mathbf{I}_m)$, $m \geq 2$, and $n > m-1$. For an m-dimensional deterministic vector $\mathbf{q}$, we define $\mathbf{Q} = [\tilde{\mathbf{q}},\ \mathbf{Q}_1]$ as an $m \times m$ orthonormal matrix such that its first column is $\tilde{\mathbf{q}} = \mathbf{q}/(\mathbf{q}'\mathbf{q})^{\frac{1}{2}}$. Then, it holds that*

$$|\mathbf{W}| \stackrel{d}{=} \prod_{i=1}^{m} v_i, \tag{69}$$

$$\mathbf{q}'\mathbf{W}^{-1}\mathbf{q} \stackrel{d}{=} \frac{\mathbf{q}'\mathbf{q}}{v_1}, \tag{70}$$

$$\mathbf{Q}_1'\mathbf{W}^{-1}\mathbf{q} \stackrel{d}{=} \frac{\sqrt{\mathbf{q}'\mathbf{q}}}{v_1\sqrt{v_2}}\mathbf{y}, \tag{71}$$

where $v_i \sim \chi^2_{n-m+i}$, $i = 1, \ldots, m$, $\mathbf{y} \sim N_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$, and they are mutually independent.

Additionally, let $\mathbf{q}$ and $\boldsymbol{\mu}$ be m-dimensional deterministic vectors such that $\mathbf{q} \neq a\boldsymbol{\mu}$ for any $a \in \mathbb{R}$. Then, it holds that

$$\mathbf{q}'\mathbf{W}^{-1}\boldsymbol{\mu} \stackrel{d}{=} \frac{1}{v_1}\left(\mathbf{q}'\boldsymbol{\mu} + \frac{\sqrt{c}y_1}{\sqrt{v_2}}\right), \tag{72}$$

where y_1 is the first element of $\mathbf{y}$ and $c = (\mathbf{q}'\mathbf{q})(\boldsymbol{\mu}'\boldsymbol{\mu}) - (\mathbf{q}'\boldsymbol{\mu})^2$.

Proof of Lemma 5.2: Let

$$\mathbf{A} = \begin{bmatrix} A_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = (\mathbf{Q}'\mathbf{W}^{-1}\mathbf{Q})^{-1}, \tag{73}$$

with A_{11} univariate. Theorem 3.2.11 in Muirhead [1982] implies that $\mathbf{A} = (\mathbf{Q}'\mathbf{W}^{-1}\mathbf{Q})^{-1} \sim W_m(n, \mathbf{I}_m)$, and Theorem 3.2.10 of Muirhead [1982] yields

$$A_{11\cdot 2} = A_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} \equiv v_1 \sim \chi^2_{n-m+1}, \tag{74}$$

$$-\mathbf{A}_{22}^{-\frac{1}{2}}\mathbf{A}_{21} \equiv \mathbf{x} \sim N_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1}), \tag{75}$$

$$\mathbf{A}_{22} \sim W_{m-1}(n, \mathbf{I}_{m-1}), \tag{76}$$

where v_1, $\mathbf{x}$ and $\mathbf{A}_{22}$ are mutually independent. In the derivation of (75), it is used that $\mathbf{A}_{21}|\mathbf{A}_{22} \sim N_{m-1}(\mathbf{0}_{m-1}, \mathbf{A}_{22})$ and, consequently, $\mathbf{x} = \mathbf{A}_{22}^{-\frac{1}{2}}\mathbf{A}_{21}$ is unconditionally standard multivariate normally distributed, and it is independent of $\mathbf{A}_{22}$.

Hence, it follows that

$$|\mathbf{W}| = |\mathbf{A}| = A_{11\cdot 2}|\mathbf{A}_{22}| \stackrel{d}{=} v_1|\mathbf{A}_{22}|, \tag{77}$$

$$\mathbf{q}'\mathbf{W}^{-1}\mathbf{q} = (\mathbf{q}'\mathbf{q})\tilde{\mathbf{q}}'\mathbf{W}^{-1}\tilde{\mathbf{q}} = (\mathbf{q}'\mathbf{q})A_{11\cdot 2}^{-1} \stackrel{d}{=} \frac{\mathbf{q}'\mathbf{q}}{v_1}, \tag{78}$$

$$\mathbf{Q}_1'\mathbf{W}^{-1}\mathbf{q} = (\mathbf{q}'\mathbf{q})^{\frac{1}{2}}\mathbf{Q}_1'\mathbf{W}^{-1}\tilde{\mathbf{q}} = -(\mathbf{q}'\mathbf{q})^{\frac{1}{2}}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}A_{11\cdot 2}^{-1} \stackrel{d}{=} \frac{\sqrt{\mathbf{q}'\mathbf{q}}\mathbf{A}_{22}^{-\frac{1}{2}}\mathbf{x}}{v_1}. \tag{79}$$

Since v_1, $\mathbf{A}_{22}^{-\frac{1}{2}}\mathbf{x}$, and $\mathbf{A}_{22}$ are mutually independent with $\mathbf{x} \sim N_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$ and $\mathbf{A}_{22} \sim W_{m-1}(n, \mathbf{I}_{m-1})$, the application of Corollary 5.1 completes the proof of (69), (70), and (71) when $m > 2$. For $m = 2$, equation (71) follows from the fact that A_{22} is univariate and it is χ^2-distributed with n degrees of freedom.

To derive (72), we set the first column of $\mathbf{Q}_1$ equal to

$$\boldsymbol{\xi} = \frac{(\mathbf{I}_m - \tilde{\mathbf{q}}\tilde{\mathbf{q}}')\boldsymbol{\mu}}{[\boldsymbol{\mu}'(\mathbf{I}_m - \tilde{\mathbf{q}}\tilde{\mathbf{q}}')\boldsymbol{\mu}]^{\frac{1}{2}}}. \tag{80}$$

Then,

$$\mathbf{q}'\mathbf{W}^{-1}\boldsymbol{\xi} = (\mathbf{q}'\mathbf{q})^{\frac{1}{2}}\tilde{\mathbf{q}}'\mathbf{W}^{-1}\mathbf{Q}_1\begin{bmatrix} 1 \\ \mathbf{0}_{m-2} \end{bmatrix} \stackrel{d}{=} \frac{(\mathbf{q}'\mathbf{q})^{\frac{1}{2}}y_1}{v_1\sqrt{v_2}}. \tag{81}$$

On the other side, we get

$$\mathbf{q}'\mathbf{W}^{-1}\boldsymbol{\xi} = \frac{\mathbf{q}'\mathbf{W}^{-1}(\mathbf{I}_m - \tilde{\mathbf{q}}\tilde{\mathbf{q}}')\boldsymbol{\mu}}{[\boldsymbol{\mu}'\boldsymbol{\mu} - (\mathbf{q}'\boldsymbol{\mu})^2/(\mathbf{q}'\mathbf{q})]^{\frac{1}{2}}} = \frac{\mathbf{q}'\mathbf{W}^{-1}\boldsymbol{\mu} - (\tilde{\mathbf{q}}'\mathbf{W}^{-1}\tilde{\mathbf{q}})\mathbf{q}'\boldsymbol{\mu}}{(\mathbf{q}'\mathbf{q})^{-\frac{1}{2}}\sqrt{c}}, \tag{82}$$

and, consequently,

$$\mathbf{q}'\mathbf{W}^{-1}\boldsymbol{\mu} = (\mathbf{q}'\mathbf{q})^{-\frac{1}{2}}\sqrt{c}\mathbf{q}'\mathbf{W}^{-1}\boldsymbol{\xi} + (\tilde{\mathbf{q}}'\mathbf{W}^{-1}\tilde{\mathbf{q}})\mathbf{q}'\boldsymbol{\mu} \stackrel{d}{=} \frac{1}{v_1}\left(\mathbf{q}'\boldsymbol{\mu} + \frac{\sqrt{c}y_1}{\sqrt{v_2}}\right). \tag{83}$$

This completes the proof of the lemma. $\square$

5.2 Density of a quadratic form in standard normal random variables

Suppose $\mathbf{x} \sim N_m(\mathbf{0}_m, \mathbf{I}_m)$. We are interested in obtaining the density of $Q = a_0 + \mathbf{a}_1'\mathbf{x} + \mathbf{x}'\mathbf{D}\mathbf{x}$, where $\mathbf{D} = \mathrm{Diag}(d_1, \ldots, d_m)$, and $d_1 \geq d_2 \cdots \geq d_m > 0$. Note that we can write

$$Q = (\mathbf{x} - \boldsymbol{\mu})'\mathbf{D}(\mathbf{x} - \boldsymbol{\mu}) + a_0 - \boldsymbol{\mu}'\mathbf{D}\boldsymbol{\mu} = \tilde{Q} + a_0 - \boldsymbol{\mu}'\mathbf{D}\boldsymbol{\mu}, \tag{84}$$

where $\tilde{Q} = (\mathbf{x} - \boldsymbol{\mu})'\mathbf{D}(\mathbf{x} - \boldsymbol{\mu})$ and $\boldsymbol{\mu} = -\frac{1}{2}\mathbf{D}^{-1}\mathbf{a}_1$. It follows that

$$f_Q(q) = f_{\tilde{Q}}(\tilde{q}), \tag{85}$$

where $\tilde{q} = q - a_0 + \boldsymbol{\mu}'\mathbf{D}\boldsymbol{\mu}$. As a result, we only need to compute the density of $\tilde{Q}$. Note that we can also write

$$\tilde{Q} = \sum_{i=1}^{m} d_i X_i, \tag{86}$$

where $X_i \sim \chi_1^2(\delta_i)$, $i = 1, \ldots, m$, are independent noncentral chi-squared random variables with 1 degree of freedom and a noncentrality parameter of

$$\delta_i = \frac{a_{1i}^2}{4d_i^2}. \tag{87}$$

There are two different approaches for computing the density of $\tilde{Q}$. The first is an integration approach that is based on the inversion of the characteristic function of $\tilde{Q}$. It can be readily shown that

$$f_{\tilde{Q}}(\tilde{q}) = \frac{1}{\pi}\int_0^\infty \Re\left(e^{-\mathrm{i}t\tilde{q}}\varphi_{\tilde{Q}}(t)\right)\mathrm{d}t, \tag{88}$$

where $\varphi_{\tilde{Q}}(t)$ is the characteristic function of $\tilde{Q}$ and it is given by

$$\varphi_{\tilde{Q}}(t) = \exp\left(-\frac{\boldsymbol{\mu}'\boldsymbol{\mu}}{2} + \frac{\boldsymbol{\mu}'(\mathbf{I}_m - 2\mathrm{i}t\mathbf{D})^{-1}\boldsymbol{\mu}}{2}\right)|\mathbf{I}_m - 2\mathrm{i}t\mathbf{D}|^{-\frac{1}{2}}. \tag{89}$$

Alternatively, we can directly use the characteristic function of Q to obtain

$$f_Q(q) = \frac{1}{\pi}\int_0^\infty \Re\left(e^{-\mathrm{i}tq}\varphi_Q(t)\right)\mathrm{d}t, \tag{90}$$

where $\varphi_Q(t)$ is the characteristic function of Q and it is given by

$$\varphi_Q(t) = \exp\left(\mathrm{i}ta_0 - \frac{t^2\mathbf{a}_1'(\mathbf{I}_m - 2\mathrm{i}t\mathbf{D})^{-1}\mathbf{a}_1}{2}\right)|\mathbf{I}_m - 2\mathrm{i}t\mathbf{D}|^{-\frac{1}{2}}. \tag{91}$$

A second approach is to write the density of $\tilde{Q}$ as a mixture of chi-squared density. Ruben [1962] shows that for any $\beta > 0$, we have

$$f_{\tilde{Q}}(\tilde{q}) = \frac{1}{\beta}\sum_{i=0}^{\infty} w_i f_{m+2i}\left(\frac{\tilde{q}}{\beta}\right), \tag{92}$$

where $f_\nu(x)$ is the density function of χ_ν^2, and w_i can be recursively obtained using

$$w_0 = \exp\left(-\frac{1}{2}\sum_{j=1}^{m}\delta_j\right)\prod_{j=1}^{m}\left(\frac{\beta}{d_j}\right)^{\frac{1}{2}}, \tag{93}$$

$$w_{i+1} = \frac{1}{2i}\sum_{k=0}^{i} w_i p_{k-i}, \quad i > 0, \tag{94}$$

and

$$p_i = \sum_{j=1}^{m}\left(1 - \frac{\beta}{d_j}\right)^i + i\sum_{j=1}^{m}\frac{\beta\delta_j}{d_j}\left(1 - \frac{\beta}{d_j}\right)^{i-1}. \tag{95}$$

Note that for any $\beta > 0$, $\sum_{i=0}^{\infty} w_i = 1$. A sufficient condition for $w_i > 0$ is $\beta \leq d_m$.

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