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Abstract

We derive the exact density function of the product of an inverse Wishart random matrix and an independent normal random vector. The density is expressed as a one-dimensional integral that can be evaluated quickly and accurately by standard quadrature methods. We further obtain integral representations for the density of linear combinations of the elements of this product; in particular, the joint density of any p linear combinations can be recovered from an integral of dimension at most $\min(p+1,\ m-p+1,\ 6)$, where m is the dimension of the random vector. All results are established in the general setting in which the scale matrix of the Wishart distribution and the covariance matrix of the normal vector are arbitrary positive definite matrices, and simplified formulas involving integrals of dimension at most three are provided for the important special case in which the two covariance matrices are proportional.

Keywords: Inverse Wishart distribution; multivariate normal distribution; exact distribution; integral representation; hypergeometric function; parabolic cylinder function

1 Introduction

The assumption of multivariate normality underpins a wide range of classical and modern statistical procedures. When the data $\mathbf{x}_1, \ldots, \mathbf{x}_N$ constitute a sample from an m-variate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, denoted by $\mathcal{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, inference about the parameters is typically based on the sample mean vector and the sample covariance matrix,

$$\hat{\boldsymbol{\mu}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i \qquad \text{and} \qquad \hat{\boldsymbol{\Sigma}} = \frac{1}{N-1}\sum_{i=1}^{N}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})'. \tag{1}$$

Under the normal model, these estimators have remarkably tractable distributions: the sample mean is normally distributed, $\hat{\boldsymbol{\mu}} \sim \mathcal{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}/N)$, the scaled sample covariance matrix follows a Wishart distribution, $(N-1)\hat{\boldsymbol{\Sigma}} \sim \mathcal{W}_m(N-1, \boldsymbol{\Sigma})$, and the two estimators are independent (Muirhead, 1982, Chapter 3).

The marginal distributions of $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ have been studied extensively. Work on the mean estimator includes shrinkage methods (Bodnar et al., 2019b), improved risk properties (Jorion, 1986), and optimal confidence regions (Efron, 2006). Research on covariance and precision matrices ranges from classical distributional analysis of Wishart and inverse Wishart matrices (Haff, 1979; Srivastava, 2003; Kubokawa and Srivastava, 2008; Bodnar and Okhrin, 2008; Cook and Forzani, 2011; Bodnar et al., 2019a; Imori and von Rosen, 2020; Kozubowski et al., 2025) to modern shrinkage and regularization approaches tailored to high-dimensional settings (Wang et al., 2015; Ledoit and Wolf, 2020, 2022; Bodnar et al., 2022a; Bodnar and Parolya, 2026). Both strands of the literature are rich and well developed.

By contrast, considerably less is known about quantities in which the sample mean vector and the sample covariance matrix appear *jointly*, even though both estimators frequently enter the same statistical procedure. This raises the question of how to characterize the distribution of functions that combine a normal vector with a Wishart-type matrix. In recent years, several authors have addressed this problem from complementary angles. Exact and asymptotic distributions have been established for products of an inverse Wishart matrix and a normal vector (Bodnar and Okhrin, 2011; Kotsiuba and Mazur, 2015), and parallel results have been obtained for the corresponding Wishart–normal products (Bodnar et al., 2013, 2014, 2018; Yonenaga and Suzukawa, 2023; Bodnar et al., 2026a). Such products arise in a variety of applications. Inverse Wishart–normal combinations underpin the coefficients of classical discriminant rules (Muirhead, 1982; Rencher and Christensen, 2012; Bodnar et al., 2020) and the weights of estimated optimal portfolios (Jobson and Korkie, 1980; Kan and Zhou, 2007; Kan and Smith, 2008; Bodnar and Schmid, 2009; Javed et al., 2021; Karlsson et al., 2021; Kan et al., 2022; Bodnar et al., 2022b, 2023; Kan et al., 2024a; Kan and Lassance, 2025), whereas Wishart–normal products appear in Bayesian analyses built on conjugate normal–inverse-Wishart priors for the mean vector and covariance matrix (Bernardo and Smith, 1994).

The main objective of this paper is to extend the results of Bodnar and Okhrin (2011) on the product of an inverse Wishart matrix and a normal vector to a substantially more flexible framework in which the two covariance matrices are allowed to differ. Specifically, let $\mathbf{W} \sim \mathcal{W}_m(n, \boldsymbol{\Sigma}_w)$ be an $m \times m$ Wishart matrix with n degrees of freedom and scale matrix $\boldsymbol{\Sigma}_w$, and let $\mathbf{z} \sim \mathcal{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}_z)$ be an m-dimensional normal vector with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}_z$, independent of $\mathbf{W}$. Throughout, $\boldsymbol{\Sigma}_w$ and $\boldsymbol{\Sigma}_z$ are positive definite and $n > m - 1$, so that $\mathbf{W}$ is nonsingular with probability one. We derive the exact density of the product $\mathbf{W}^{-1}\mathbf{z}$, which, to the best of our knowledge, is new to the literature, and we obtain integral representations for the density of $\mathbf{L}'\mathbf{W}^{-1}\mathbf{z}$, where $\mathbf{L}$ is an $m \times p$ matrix of constants with full column rank $p \leq m$. The practically important case of a single linear combination, $p = 1$ with $\mathbf{L} = \boldsymbol{\alpha}$, receives special attention. Our contribution relative to Bodnar and Okhrin (2011) is thus twofold. First, we dispense with the assumption that the two covariance matrices coincide. Second, even within the equal (or, more generally, proportional) covariance setting that they study, we go beyond the single linear combination treated there: we provide the joint density of the full vector $\mathbf{W}^{-1}\mathbf{z}$ as a one-dimensional integral, and the joint density of any number $p \leq m$ of linear combinations through integral representations of dimension at most three; see Section 4 for a detailed account.

The remainder of the paper is organized as follows. Section 2 derives the density of $\mathbf{W}^{-1}\mathbf{z}$. Section 3 treats linear combinations $\mathbf{L}'\mathbf{W}^{-1}\mathbf{z}$ and provides a numerical illustration. Section 4 presents the simplifications that become available when the two covariance matrices are proportional. Section 5 concludes. Technical lemmas and their proofs are collected in the Appendix.

2 The density of the product of an inverse Wishart matrix and a normal vector

Let $\mathbf{W} \sim \mathcal{W}_m(n, \boldsymbol{\Sigma}_w)$ and $\mathbf{z} \sim \mathcal{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}_z)$ be independent, where $\boldsymbol{\Sigma}_w$ and $\boldsymbol{\Sigma}_z$ are positive definite and $n > m-1$. Following Muirhead (1982, p. 87), the degrees of freedom n need not be an integer; any real $n > m-1$ is admissible. Throughout the paper, $f_\nu(\cdot)$ denotes the density function of a χ^2_ν random variable, and for a real number a and a positive integer k we write $(a)_k = \prod_{i=0}^{k-1}(a+i)$ for the ascending factorial. For a positive definite matrix $\mathbf{A}$, $\mathbf{A}^{\frac{1}{2}}$ denotes its unique symmetric positive definite square root and $\mathbf{A}^{-\frac{1}{2}} = (\mathbf{A}^{\frac{1}{2}})^{-1}$. This section derives the density function of $\mathbf{q} = \mathbf{W}^{-1}\mathbf{z}$; linear combinations of the elements of $\mathbf{q}$ are treated in Section 3.

Consider first the univariate case $m = 1$, with $\Sigma_w = \sigma_w^2$ and $\Sigma_z = \sigma_z^2$. Then $W = \sigma_w^2 w$ with $w \sim \chi^2_n$, and conditioning on w yields

$$f_q(q) = \mathbb{E}[f(q|W)] = \int_0^\infty \frac{\sigma_w^2 w}{\sqrt{2\pi}\,\sigma_z} \exp\left(-\frac{(q\sigma_w^2 w - \mu)^2}{2\sigma_z^2}\right) f_n(w)\,\mathrm{d}w, \tag{2}$$

so the density of q is available as a single integral. The following theorem shows that the same is true for every $m \geq 2$: despite the matrix inversion, the density of $\mathbf{q}$ can always be written as a one-dimensional integral. For $m = 1$ every linear combination is a scalar multiple of q, so the scalar case is covered by (2); accordingly, all the theorems below are stated for $m \geq 2$.

Theorem 1. *Let $\mathbf{W} \sim \mathcal{W}_m(n, \boldsymbol{\Sigma}_w)$ and $\mathbf{z} \sim \mathcal{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}_z)$ be independent, with $m \geq 2$, $n > m-1$, and positive definite $\boldsymbol{\Sigma}_w$ and $\boldsymbol{\Sigma}_z$. Define*

$$\hat{\boldsymbol{\mu}} = \boldsymbol{\Sigma}_w^{\frac{1}{2}}\boldsymbol{\Sigma}_z^{-1}\boldsymbol{\mu}, \qquad \hat{\mathbf{q}} = \boldsymbol{\Sigma}_w^{\frac{1}{2}}\mathbf{q}, \qquad \tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\boldsymbol{\Sigma}_z\boldsymbol{\Sigma}_w^{-\frac{1}{2}}, \tag{3}$$

and, for $\mathbf{q} \neq \mathbf{0}_m$,

$$\tilde{\mathbf{q}} = \frac{\hat{\mathbf{q}}}{(\hat{\mathbf{q}}'\hat{\mathbf{q}})^{\frac{1}{2}}}, \qquad \boldsymbol{\xi} = \frac{(\mathbf{I}_m - \tilde{\mathbf{q}}\tilde{\mathbf{q}}')\hat{\boldsymbol{\mu}}}{[\hat{\boldsymbol{\mu}}'(\mathbf{I}_m - \tilde{\mathbf{q}}\tilde{\mathbf{q}}')\hat{\boldsymbol{\mu}}]^{\frac{1}{2}}}. \tag{4}$$

Let $\mathbf{Q} = [\tilde{\mathbf{q}},\ \mathbf{Q}_1]$ be an $m \times m$ orthonormal matrix whose first column is $\tilde{\mathbf{q}}$ and such that the first column of $\mathbf{Q}_1$ equals $\boldsymbol{\xi}$ (when $(\mathbf{I}_m - \tilde{\mathbf{q}}\tilde{\mathbf{q}}')\hat{\boldsymbol{\mu}} = \mathbf{0}_m$, the constant c below equals zero and $\mathbf{Q}_1$ can be any orthonormal complement of $\tilde{\mathbf{q}}$), and partition

$$\mathbf{C} = \mathbf{Q}'\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{Q} = \begin{bmatrix} C_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}, \qquad C_{11} = \tilde{\mathbf{q}}'\tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\mathbf{q}}, \tag{5}$$

where C_{11} is a scalar. Then the density function of $\mathbf{q} = \mathbf{W}^{-1}\mathbf{z}$ is given, for $\mathbf{q} \neq \mathbf{0}_m$, by

$$f_{\mathbf{q}}(\mathbf{q}) = \frac{\exp\left(-\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}_z^{-1}\boldsymbol{\mu}}{2}\right)|\boldsymbol{\Sigma}_w|\,(n-m+1)_m}{(2\pi)^{\frac{m}{2}}|\boldsymbol{\Sigma}_z|^{\frac{1}{2}}} \times \int_0^\infty |\mathbf{I}_{m-1} + \mathbf{B}|^{-\frac{1}{2}} \exp\left(b_0 + \frac{\mathbf{b}_1'(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1}{2}\right) f_{n+2}(u_1)\,\mathrm{d}u_1, \tag{6}$$

where

$$b_0 = (\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}})u_1 - \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})u_1^2 C_{11}}{2}, \tag{7}$$

$$\mathbf{b}_1 = \sqrt{c}\sqrt{u_1}\,\mathbf{e}_1 - (\hat{\mathbf{q}}'\hat{\mathbf{q}})u_1^{\frac{3}{2}}\mathbf{C}_{21}, \tag{8}$$

$$\mathbf{B} = (\hat{\mathbf{q}}'\hat{\mathbf{q}})u_1\mathbf{C}_{22}, \tag{9}$$

with $c = (\hat{\mathbf{q}}'\hat{\mathbf{q}})(\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}) - (\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}})^2$ *and* $\mathbf{e}_1 = [1,\ \mathbf{0}'_{m-2}]'$*; at the origin,*

$$f_{\mathbf{q}}(\mathbf{0}_m) = \frac{\exp\left(-\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}_z^{-1}\boldsymbol{\mu}}{2}\right)|\boldsymbol{\Sigma}_w|\,(n-m+1)_m}{(2\pi)^{\frac{m}{2}}|\boldsymbol{\Sigma}_z|^{\frac{1}{2}}}. \tag{10}$$

Proof. Let $\tilde{\mathbf{W}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\mathbf{W}\boldsymbol{\Sigma}_w^{-\frac{1}{2}} \sim \mathcal{W}_m(n, \mathbf{I}_m)$. Conditional on $\tilde{\mathbf{W}}$,

$$\mathbf{q} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\tilde{\mathbf{W}}^{-1}\boldsymbol{\Sigma}_w^{-\frac{1}{2}}\mathbf{z}\,|\,\tilde{\mathbf{W}} \sim \mathcal{N}_m\left(\boldsymbol{\Sigma}_w^{-\frac{1}{2}}\tilde{\mathbf{W}}^{-1}\boldsymbol{\Sigma}_w^{-\frac{1}{2}}\boldsymbol{\mu},\ \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\tilde{\mathbf{W}}^{-1}\boldsymbol{\Sigma}_w^{-\frac{1}{2}}\boldsymbol{\Sigma}_z\boldsymbol{\Sigma}_w^{-\frac{1}{2}}\tilde{\mathbf{W}}^{-1}\boldsymbol{\Sigma}_w^{-\frac{1}{2}}\right), \tag{11}$$

so the conditional density of $\mathbf{q}$ is

$$f(\mathbf{q}|\tilde{\mathbf{W}}) = \frac{|\boldsymbol{\Sigma}_w||\tilde{\mathbf{W}}|\exp\left(-\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}_z^{-1}\boldsymbol{\mu}}{2}\right)}{(2\pi)^{\frac{m}{2}}|\boldsymbol{\Sigma}_z|^{\frac{1}{2}}}\exp\left(-\frac{\alpha_1}{2}+\alpha_2\right), \tag{12}$$

where

$$\alpha_1 = \mathbf{q}'\boldsymbol{\Sigma}_w^{\frac{1}{2}}\tilde{\mathbf{W}}\boldsymbol{\Sigma}_w^{\frac{1}{2}}\boldsymbol{\Sigma}_z^{-1}\boldsymbol{\Sigma}_w^{\frac{1}{2}}\tilde{\mathbf{W}}\boldsymbol{\Sigma}_w^{\frac{1}{2}}\mathbf{q} = \hat{\mathbf{q}}'\tilde{\mathbf{W}}\tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\mathbf{W}}\hat{\mathbf{q}}, \tag{13}$$

$$\alpha_2 = \mathbf{q}'\boldsymbol{\Sigma}_w^{\frac{1}{2}}\tilde{\mathbf{W}}\boldsymbol{\Sigma}_w^{\frac{1}{2}}\boldsymbol{\Sigma}_z^{-1}\boldsymbol{\mu} = \hat{\mathbf{q}}'\tilde{\mathbf{W}}\hat{\boldsymbol{\mu}}. \tag{14}$$

The key observation is that $f(\mathbf{q}|\tilde{\mathbf{W}})$ depends on $\tilde{\mathbf{W}}$ only through the three random quantities α_1, α_2, and $|\tilde{\mathbf{W}}|$, for which a joint stochastic representation is available.

Consider first the special case $\mathbf{q} = \mathbf{0}_m$. Then $\alpha_1 = \alpha_2 = 0$, and using $\mathbb{E}[|\tilde{\mathbf{W}}|] = \prod_{i=1}^m (n+1-i) = (n-m+1)_m$ we obtain (10). Note that (10) also coincides with the limit of (6) as $\mathbf{q} \to \mathbf{0}_m$, in which case $c = 0$, $b_0 = 0$, $\mathbf{b}_1 = \mathbf{0}_{m-1}$, and $\mathbf{B}$ is the zero matrix, so that the integral in (6) equals one.

Now let $\mathbf{q} \neq \mathbf{0}_m$. Let $\mathbf{y} \sim \mathcal{N}_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$ and $u_i \sim \chi^2_{n+1-i}$, $i = 1, \ldots, m$, be mutually independent. Applying Lemma 1 in the Appendix with $\tilde{\mathbf{q}}$, $\boldsymbol{\xi}$, and $\mathbf{Q}$ as defined in the statement of the theorem gives

$$|\tilde{\mathbf{W}}| \stackrel{d}{=} \prod_{i=1}^m u_i, \tag{15}$$

$$\tilde{\mathbf{q}}'\tilde{\mathbf{W}}\tilde{\mathbf{q}} \stackrel{d}{=} u_1, \tag{16}$$

$$\mathbf{Q}_1'\tilde{\mathbf{W}}\tilde{\mathbf{q}} \stackrel{d}{=} \sqrt{u_1}\,\mathbf{y}, \tag{17}$$

$$\alpha_2 \stackrel{d}{=} (\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}})u_1 + \sqrt{c}\sqrt{u_1}\,y_1, \tag{18}$$

where y_1 is the first element of $\mathbf{y}$ and $c = (\hat{\mathbf{q}}'\hat{\mathbf{q}})(\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}) - (\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}})^2$. Moreover, from the definition of $\mathbf{C}$ in (5),

$$\alpha_1 = (\hat{\mathbf{q}}'\hat{\mathbf{q}})\,\tilde{\mathbf{q}}'\tilde{\mathbf{W}}\mathbf{Q}\mathbf{Q}'\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{Q}\mathbf{Q}'\tilde{\mathbf{W}}\tilde{\mathbf{q}} = (\hat{\mathbf{q}}'\hat{\mathbf{q}})\,\mathbf{h}'\mathbf{C}\mathbf{h}, \qquad \mathbf{h} = \mathbf{Q}'\tilde{\mathbf{W}}\tilde{\mathbf{q}} = \begin{bmatrix} u_1 \\ \sqrt{u_1}\,\mathbf{y} \end{bmatrix}. \tag{19}$$

Since $u_1, \ldots, u_m$ and $\mathbf{y}$ are mutually independent and

$$\mathbb{E}\left[\prod_{i=2}^m u_i\right] = \prod_{i=2}^m (n+1-i) = (n-m+1)_{m-1}, \tag{20}$$

we obtain

$$f_{\mathbf{q}}(\mathbf{q}) = \mathbb{E}[f(\mathbf{q}|\tilde{\mathbf{W}})] = \frac{\exp\left(-\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}_z^{-1}\boldsymbol{\mu}}{2}\right)|\boldsymbol{\Sigma}_w|\,(n-m+1)_{m-1}}{(2\pi)^{\frac{m}{2}}|\boldsymbol{\Sigma}_z|^{\frac{1}{2}}}\,\mathbb{E}\left[u_1\exp\left(-\frac{\alpha_1}{2}+\alpha_2\right)\right]. \tag{21}$$

Conditional on u_1, α_1 is a quadratic function of $\mathbf{y}$ and α_2 is a linear function of $\mathbf{y}$:

$$\alpha_1 = (\hat{\mathbf{q}}'\hat{\mathbf{q}})u_1^2 C_{11} + 2u_1^{\frac{3}{2}}(\hat{\mathbf{q}}'\hat{\mathbf{q}})\mathbf{C}_{21}'\mathbf{y} + u_1(\hat{\mathbf{q}}'\hat{\mathbf{q}})\mathbf{y}'\mathbf{C}_{22}\mathbf{y}, \tag{22}$$

$$\alpha_2 = (\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}})u_1 + \sqrt{c}\sqrt{u_1}\,\mathbf{e}_1'\mathbf{y}. \tag{23}$$

Hence

$$\mathbb{E}\left[u_1 \exp\left(-\frac{\alpha_1}{2} + \alpha_2\right)\right] = \mathbb{E}\left[u_1 \exp\left(b_0 + \mathbf{b}_1'\mathbf{y} - \frac{\mathbf{y}'\mathbf{B}\mathbf{y}}{2}\right)\right], \tag{24}$$

with b_0, $\mathbf{b}_1$, and $\mathbf{B}$ given in (7)–(9). Evaluating the inner expectation over $\mathbf{y}$ by Lemma 2 in the Appendix yields

$$f_{\mathbf{q}}(\mathbf{q}) = \frac{\exp\left(-\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}_z^{-1}\boldsymbol{\mu}}{2}\right)|\boldsymbol{\Sigma}_w|\,(n-m+1)_{m-1}}{(2\pi)^{\frac{m}{2}}|\boldsymbol{\Sigma}_z|^{\frac{1}{2}}} \times \int_0^\infty |\mathbf{I}_{m-1}+\mathbf{B}|^{-\frac{1}{2}} \exp\left(b_0 + \frac{\mathbf{b}_1'(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1}{2}\right) u_1 f_n(u_1)\,\mathrm{d}u_1. \tag{25}$$

Finally, using the identity $u_1 f_n(u_1) = n f_{n+2}(u_1)$ together with $(n-m+1)_{m-1}\,n = (n-m+1)_m$ delivers (6). □

Theorem 1 shows that, just as in the univariate case, the density of $\mathbf{q}$ can be computed by one-dimensional numerical integration for every m. The evaluation of the integrand can be organized efficiently through a spectral decomposition. Let $\mathbf{C}_{22} = \mathbf{P}\mathbf{D}\mathbf{P}'$, where $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_{m-1})$ contains the eigenvalues of $\mathbf{C}_{22} = \mathbf{Q}_1'\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{Q}_1$, and $\mathbf{P}$ contains the corresponding eigenvectors. Then

$$|\mathbf{I}_{m-1}+\mathbf{B}| = \prod_{i=1}^{m-1}\left(1 + (\hat{\mathbf{q}}'\hat{\mathbf{q}})u_1 d_i\right), \tag{26}$$

$$\mathbf{b}_1'(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1 = \sum_{i=1}^{m-1}\frac{\tilde{b}_{1i}^2}{1 + (\hat{\mathbf{q}}'\hat{\mathbf{q}})u_1 d_i}, \tag{27}$$

where $\tilde{\mathbf{b}}_1 = \mathbf{P}'\mathbf{b}_1$. Since the decomposition does not depend on u_1, it needs to be computed only once, after which each evaluation of the integrand requires just $O(m)$ operations. This avoids repeated matrix inversions and enhances numerical robustness.

3 Linear combinations of the product

In applications one is frequently interested not in the full vector $\mathbf{W}^{-1}\mathbf{z}$ but in a small number of linear combinations of its elements. We therefore consider $\mathbf{q}_\mathbf{L} = \mathbf{L}'\mathbf{W}^{-1}\mathbf{z}$, where $\mathbf{L}$ is an $m \times p$ matrix of constants with full column rank $p \leq m$, and we study in detail the special case $p = 1$, in which $\mathbf{L} = \boldsymbol{\alpha}$ is an m-dimensional vector.

Theorem 2. *Let $\mathbf{W} \sim \mathcal{W}_m(n, \boldsymbol{\Sigma}_w)$ and $\mathbf{z} \sim \mathcal{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}_z)$ be independent, with $m \geq 2$, $n > m-1$, and positive definite $\boldsymbol{\Sigma}_w$ and $\boldsymbol{\Sigma}_z$. Let $\mathbf{L}$ be an $m \times p$ matrix of constants with rank $p \leq m$, and define $\tilde{\mathbf{L}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\mathbf{L}$ and $\tilde{\mathbf{z}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\mathbf{z}$. Then the density function of $\mathbf{q}_\mathbf{L} = \mathbf{L}'\mathbf{W}^{-1}\mathbf{z}$ is given as follows.*

(i) For $p = m$:

$$f_{\mathbf{q}_\mathbf{L}}(\mathbf{q}_\mathbf{L}) = |\mathbf{L}'\mathbf{L}|^{-\frac{1}{2}} f_{\mathbf{q}}\left((\mathbf{L}^{-1})'\mathbf{q}_\mathbf{L}\right), \tag{28}$$

where $f_{\mathbf{q}}(\cdot)$ is given in Theorem 1.

(ii) For $1 < p < m$:

$$f_{\mathbf{q_L}}(\mathbf{q_L}) = \frac{2^p\,\Gamma\left(\frac{n-m+2+p}{2}\right)\left(\frac{n-m+1}{2}\right)_p}{\Gamma\left(\frac{n-m+2}{2}\right)\pi^{\frac{p}{2}}|\tilde{\mathbf{L}}'\tilde{\mathbf{L}}|^{\frac{1}{2}}} \times \int_0^\infty\int_0^\infty\int_0^\infty\int_{-\infty}^\infty \frac{f_{u,v,\hat{u}}(u,v,\hat{u})\,f_{n-m+1+2p}(v_1)}{(u+\hat{u})^{\frac{p-1}{2}}\sqrt{\hat{u}}\,(1+a)^{\frac{n-m+2+p}{2}}}\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}\hat{u}\,\mathrm{d}v_1, \tag{29}$$

where $f_{u,v,\hat{u}}(u,v,\hat{u})$ is the joint density function of $u = \hat{\mathbf{z}}'\hat{\mathbf{z}}$, $v = \hat{\mathbf{q}}'\hat{\mathbf{z}}$, and $\hat{u}$, with $\hat{\mathbf{q}} = (\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-\frac{1}{2}}\mathbf{q_L}$,

$$\hat{\mathbf{z}} = (\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-\frac{1}{2}}\tilde{\mathbf{L}}'\tilde{\mathbf{z}}, \tag{30}$$

$$\hat{u} = \tilde{\mathbf{z}}'(\mathbf{I}_m - \tilde{\mathbf{L}}(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-1}\tilde{\mathbf{L}}')\tilde{\mathbf{z}}, \tag{31}$$

$$a = \frac{v_1^2}{u+\hat{u}}\left(\hat{\mathbf{q}}'\hat{\mathbf{q}} + \frac{v^2}{\hat{u}}\right) - \frac{2v_1 v}{\hat{u}} + \frac{u}{\hat{u}}. \tag{32}$$

(iii) For $p = 1$ with $\mathbf{L} = \boldsymbol{\alpha}$:

$$f_{q_{\boldsymbol{\alpha}}}(q_{\boldsymbol{\alpha}}) = \frac{(n-m+1)\,\Gamma\left(\frac{n-m+3}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-m+2}{2}\right)(\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w^{-1}\boldsymbol{\alpha})^{\frac{1}{2}}}\int_0^\infty \mathbb{E}\left[\left.\frac{\hat{u}^{\frac{n-m+2}{2}}}{(\hat{y}^2+\hat{u})^{\frac{n-m+3}{2}}}\right| v_1\right] f_{n-m+3}(v_1)\,\mathrm{d}v_1, \tag{33}$$

where the expectation is taken with respect to the conditional distribution, given v_1, of

$$\hat{u} = \mathbf{z}'\boldsymbol{\Sigma}_w^{-1}\mathbf{z} - \frac{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w^{-1}\mathbf{z})^2}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w^{-1}\boldsymbol{\alpha}}, \tag{34}$$

which is the specialization of (31) to $p = 1$ (with $\tilde{\mathbf{L}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\boldsymbol{\alpha}$), and of

$$\hat{y} = \hat{z} - v_1\hat{q} = \frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w^{-1}\mathbf{z}}{\sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w^{-1}\boldsymbol{\alpha}}} - \frac{v_1\,q_{\boldsymbol{\alpha}}}{\sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w^{-1}\boldsymbol{\alpha}}}. \tag{35}$$

Proof. (i) When $p = m$, the map $\mathbf{q} \mapsto \mathbf{L}'\mathbf{q}$ is a nonsingular linear transformation with Jacobian $|\mathbf{L}'\mathbf{L}|^{-1/2}$, and (28) follows immediately from Theorem 1.

(ii) Suppose $p < m$, and let $\tilde{\mathbf{W}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\mathbf{W}\boldsymbol{\Sigma}_w^{-\frac{1}{2}} \sim \mathcal{W}_m(n, \mathbf{I}_m)$, so that $\mathbf{q_L} = \mathbf{L}'\mathbf{W}^{-1}\mathbf{z} = \tilde{\mathbf{L}}'\tilde{\mathbf{W}}^{-1}\tilde{\mathbf{z}}$. Write $\bar{\mathbf{z}} = \tilde{\mathbf{z}}/(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{1/2}$, and let $\mathbf{Q}$ be an $m \times (m-1)$ orthonormal matrix whose columns are orthogonal to $\tilde{\mathbf{z}}$. Since $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{z}}$ are independent, applying Lemma 3 in the Appendix conditionally on $\tilde{\mathbf{z}}$ (with $\mathbf{q} = \bar{\mathbf{z}}$) yields

$$\bar{\mathbf{z}}'\tilde{\mathbf{W}}^{-1}\bar{\mathbf{z}} \overset{d}{=} \frac{1}{v_1}, \qquad \mathbf{Q}'\tilde{\mathbf{W}}^{-1}\bar{\mathbf{z}} \overset{d}{=} \frac{\mathbf{x}}{v_1\sqrt{v_2}}, \tag{36}$$

where $\mathbf{x} \sim \mathcal{N}_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$, $v_1 \sim \chi^2_{n-m+1}$, and $v_2 \sim \chi^2_{n-m+2}$ are mutually independent; because their joint distribution does not depend on the conditioning value of $\tilde{\mathbf{z}}$, they are also independent of $\tilde{\mathbf{z}}$. Consequently,

$$\mathbf{q_L} = [\tilde{\mathbf{L}}'\tilde{\mathbf{z}},\ (\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{\frac{1}{2}}\tilde{\mathbf{L}}'\mathbf{Q}]\begin{bmatrix}\bar{\mathbf{z}}'\tilde{\mathbf{W}}^{-1}\bar{\mathbf{z}}\\ \mathbf{Q}'\tilde{\mathbf{W}}^{-1}\bar{\mathbf{z}}\end{bmatrix} \overset{d}{=} \frac{\tilde{\mathbf{L}}'\tilde{\mathbf{z}}}{v_1} + \frac{(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{\frac{1}{2}}}{v_1\sqrt{v_2}}\tilde{\mathbf{L}}'\mathbf{Q}\mathbf{x}. \tag{37}$$

Setting $\mathbf{A} = \tilde{\mathbf{L}}'\mathbf{Q}\mathbf{Q}'\tilde{\mathbf{L}}$ and $\mathbf{x}_0 = \mathbf{A}^{-\frac{1}{2}}\tilde{\mathbf{L}}'\mathbf{Q}\mathbf{x} \sim \mathcal{N}_p(\mathbf{0}_p, \mathbf{I}_p)$, we can write

$$\mathbf{q_L} \overset{d}{=} \frac{\tilde{\mathbf{L}}'\tilde{\mathbf{z}}}{v_1} + \frac{(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{\frac{1}{2}}}{v_1\sqrt{v_2}}\mathbf{A}^{\frac{1}{2}}\mathbf{x}_0, \tag{38}$$

so that, conditional on $(\tilde{\mathbf{z}}, v_1, v_2)$,

$$f(\mathbf{q_L}|\tilde{\mathbf{z}}, v_1, v_2) = \frac{1}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Omega}|^{\frac{1}{2}}} \exp\left(-\frac{(\mathbf{q_L}-\boldsymbol{\eta})'\boldsymbol{\Omega}^{-1}(\mathbf{q_L}-\boldsymbol{\eta})}{2}\right), \tag{39}$$

where

$$\boldsymbol{\eta} = \frac{\tilde{\mathbf{L}}'\tilde{\mathbf{z}}}{v_1} \qquad \text{and} \qquad \boldsymbol{\Omega} = \frac{\tilde{\mathbf{z}}'\tilde{\mathbf{z}}}{v_1^2 v_2}\,\mathbf{A}. \tag{40}$$

We now show that this conditional density depends on $\tilde{\mathbf{z}}$ only through the three scalars $u = \hat{\mathbf{z}}'\hat{\mathbf{z}}$, $v = \hat{\mathbf{q}}'\hat{\mathbf{z}}$, and $\hat{u}$ defined in (30)–(31). Since $\mathbf{Q}\mathbf{Q}' = \mathbf{I}_m - \tilde{\mathbf{z}}(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{-1}\tilde{\mathbf{z}}'$, we have

$$\mathbf{A} = (\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{\frac{1}{2}}\left(\mathbf{I}_p - \frac{\hat{\mathbf{z}}\hat{\mathbf{z}}'}{\tilde{\mathbf{z}}'\tilde{\mathbf{z}}}\right)(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{\frac{1}{2}}, \tag{41}$$

and using $\tilde{\mathbf{z}}'\tilde{\mathbf{z}} = u + \hat{u}$ together with the Sherman–Morrison formula (note that $\mathbf{A}$ is almost surely nonsingular, since $\tilde{\mathbf{z}}$ lies in the column space of $\tilde{\mathbf{L}}$ with probability zero when $p < m$),

$$\boldsymbol{\Omega}^{-1} = \frac{v_1^2 v_2}{u+\hat{u}}(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-\frac{1}{2}}\left(\mathbf{I}_p + \frac{\hat{\mathbf{z}}\hat{\mathbf{z}}'}{\hat{u}}\right)(\tilde{\mathbf{L}}'\tilde{\mathbf{L}})^{-\frac{1}{2}}, \tag{42}$$

$$|\boldsymbol{\Omega}| = v_1^{-2p} v_2^{-p}(u+\hat{u})^{p-1}|\tilde{\mathbf{L}}'\tilde{\mathbf{L}}|\,\hat{u}. \tag{43}$$

It follows that

$$\theta_1 \equiv \mathbf{q}_\mathbf{L}'\boldsymbol{\Omega}^{-1}\mathbf{q_L} = \frac{v_1^2 v_2}{u+\hat{u}}\left(\hat{\mathbf{q}}'\hat{\mathbf{q}} + \frac{v^2}{\hat{u}}\right), \tag{44}$$

$$\theta_2 \equiv \mathbf{q}_\mathbf{L}'\boldsymbol{\Omega}^{-1}\boldsymbol{\eta} = \frac{v_1 v_2\, v}{\hat{u}}, \tag{45}$$

$$\theta_3 \equiv \boldsymbol{\eta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\eta} = \frac{v_2\, u}{\hat{u}}, \tag{46}$$

so that $\theta_1 - 2\theta_2 + \theta_3 = v_2\, a$ with a given in (32), and

$$f(\mathbf{q_L}|\tilde{\mathbf{z}}, v_1, v_2) = f(\mathbf{q_L}|u, v, \hat{u}, v_1, v_2) = \frac{1}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Omega}|^{\frac{1}{2}}}\exp\left(-\frac{\theta_1 - 2\theta_2 + \theta_3}{2}\right). \tag{47}$$

When $p \geq 2$, the triple $(u, v, \hat{u})$ possesses a joint density $f_{u,v,\hat{u}}$ (for $p = 1$ the pair (u, v) is degenerate, since $v^2 = \hat{q}^2 u$; that case is treated in part (iii)). Hence

$$f_{\mathbf{q_L}}(\mathbf{q_L}) = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_{-\infty}^\infty f(\mathbf{q_L}|u, v, \hat{u}, v_1, v_2)\, f_{u,v,\hat{u}}(u, v, \hat{u}) \times f_{n-m+1}(v_1) f_{n-m+2}(v_2)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}\hat{u}\,\mathrm{d}v_1\,\mathrm{d}v_2. \tag{48}$$

The integral over v_2 can be evaluated in closed form. Using

$$\int_0^\infty v_2^{\frac{p}{2}} \exp\left(-\frac{a v_2}{2}\right) f_{n-m+2}(v_2)\,\mathrm{d}v_2 = \frac{2^{\frac{p}{2}}\,\Gamma\left(\frac{n-m+2+p}{2}\right)}{(1+a)^{\frac{n-m+2+p}{2}}\,\Gamma\left(\frac{n-m+2}{2}\right)}, \tag{49}$$

we obtain

$$f(\mathbf{q_L}|u, v, \hat{u}, v_1) = \frac{\Gamma\left(\frac{n-m+2+p}{2}\right) v_1^p}{\pi^{\frac{p}{2}}\,\Gamma\left(\frac{n-m+2}{2}\right)|\tilde{\mathbf{L}}'\tilde{\mathbf{L}}|^{\frac{1}{2}}(u+\hat{u})^{\frac{p-1}{2}}\sqrt{\hat{u}}\,(1+a)^{\frac{n-m+2+p}{2}}}, \tag{50}$$

which expresses $f_{\mathbf{q}_\mathbf{L}}(\mathbf{q}_\mathbf{L})$ as a quadruple integral over $(v, u, \hat{u}, v_1)$. Finally, the identity

$$v_1^p f_{n-m+1}(v_1) = 2^p \left(\frac{n-m+1}{2}\right)_p f_{n-m+1+2p}(v_1) \tag{51}$$

gives (29).

(iii) For $p = 1$ with $\mathbf{L} = \boldsymbol{\alpha}$, the variables u and v collapse to $u = \hat{z}^2$ and $v = \hat{q}\hat{z}$, where $\hat{z}$ and $\hat{q}$ are scalars, so the representation reduces to a triple integral,

$$f_{q_{\boldsymbol{\alpha}}}(q_{\boldsymbol{\alpha}}) = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty f(q_{\boldsymbol{\alpha}}|\hat{z}, \hat{u}, v_1)\, f_{\hat{z},\hat{u}}(\hat{z}, \hat{u})\, f_{n-m+1}(v_1)\, \mathrm{d}\hat{z}\, \mathrm{d}\hat{u}\, \mathrm{d}v_1, \tag{52}$$

with

$$f(q_{\boldsymbol{\alpha}}|\hat{z}, \hat{u}, v_1) = \frac{\Gamma\left(\frac{n-m+3}{2}\right) v_1}{\sqrt{\pi}\,\Gamma\left(\frac{n-m+2}{2}\right) (\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w^{-1}\boldsymbol{\alpha})^{\frac{1}{2}} \sqrt{\hat{u}}\,(1+a)^{\frac{n-m+3}{2}}}, \qquad a = \frac{(\hat{z} - v_1\hat{q})^2}{\hat{u}} = \frac{\hat{y}^2}{\hat{u}}, \tag{53}$$

where $\hat{y}$ is defined in (35). Writing $1/[\sqrt{\hat{u}}(1+a)^{(n-m+3)/2}] = \hat{u}^{(n-m+2)/2}/(\hat{y}^2 + \hat{u})^{(n-m+3)/2}$ and taking the expectation over $(\hat{z}, \hat{u})$ conditional on v_1 yields

$$f_{q_{\boldsymbol{\alpha}}}(q_{\boldsymbol{\alpha}}) = \frac{\Gamma\left(\frac{n-m+3}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-m+2}{2}\right) (\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w^{-1}\boldsymbol{\alpha})^{\frac{1}{2}}} \int_0^\infty \mathbb{E}\left[\frac{\hat{u}^{\frac{n-m+2}{2}}}{(\hat{y}^2 + \hat{u})^{\frac{n-m+3}{2}}} \,\middle|\, v_1\right] v_1 f_{n-m+1}(v_1)\, \mathrm{d}v_1. \tag{54}$$

Using $v_1 f_{n-m+1}(v_1) = (n-m+1) f_{n-m+3}(v_1)$ delivers (33). □

The three cases of Theorem 2 differ markedly in how amenable they are to numerical evaluation. When $p = m$ (case (i)), the density reduces to the one-dimensional integral of Theorem 1, which is the most tractable scenario. For $1 < p < m$ (case (ii)), the representation is a quadruple integral involving the joint density $f_{u,v,\hat{u}}$ of two quadratic forms and one linear form in the normal vector $\tilde{\mathbf{z}}$; this joint density can itself be computed as a double integral by characteristic function inversion (Bodnar et al., 2026b, Section 5), so that substituting that representation into (29) yields a six-dimensional integration problem whose dimension does not grow with p or m. The case of a single linear combination, $p = 1$ (case (iii)), is considerably more manageable: the inner expectation can be computed as a single integral when $n - m$ is even and as a double integral when $n - m$ is odd (see Lemma 5 in the Appendix), so the density is obtained from a two- or three-dimensional integration problem overall. For small to moderate p, however, the six-dimensional route is not needed. The following proposition shows that, for every $p < m$ and arbitrary covariance matrices, a characteristic-function argument reduces the density of $\mathbf{q}_\mathbf{L}$ to an absolutely convergent $(p+1)$-dimensional integral whose integrand decays at a Gaussian rate.

Proposition 1. *Under the assumptions of Theorem 2 with $p < m$, let $\tilde{\mathbf{L}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\mathbf{L}$, $\tilde{\boldsymbol{\mu}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\boldsymbol{\mu}$, and $\tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\boldsymbol{\Sigma}_z\boldsymbol{\Sigma}_w^{-\frac{1}{2}}$, and for $\boldsymbol{\tau} \in \mathbb{R}^p$ define*

$$\mathbf{C}(\boldsymbol{\tau}) = (\boldsymbol{\tau}'\tilde{\mathbf{L}}'\tilde{\mathbf{L}}\boldsymbol{\tau})\,\mathbf{I}_m - \tilde{\mathbf{L}}\boldsymbol{\tau}\boldsymbol{\tau}'\tilde{\mathbf{L}}', \qquad \boldsymbol{\xi}(\boldsymbol{\tau}) = \tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\boldsymbol{\mu}} + \mathrm{i}\,\tilde{\mathbf{L}}\boldsymbol{\tau}, \tag{55}$$

where $\mathrm{i} = \sqrt{-1}$ denotes the imaginary unit. Then the density function of $\mathbf{q}_\mathbf{L} = \mathbf{L}'\mathbf{W}^{-1}\mathbf{z}$ is given by

$$f_{\mathbf{q}_\mathbf{L}}(\mathbf{q}_\mathbf{L}) = \frac{\left(\frac{n-m+1}{2}\right)_p}{\pi^p} \Re \int_{\mathbb{R}^p} (1 + 2\mathrm{i}\,\boldsymbol{\tau}'\mathbf{q}_\mathbf{L})^{-\frac{n-m+1+2p}{2}}\, \mathbb{E}\left[\Psi(\boldsymbol{\tau}, v_2)\right] \mathrm{d}\boldsymbol{\tau}, \tag{56}$$

where the expectation is taken with respect to $v_2 \sim \chi^2_{n-m+2}$, the principal branch of the complex power is used, the symbol $\Re$ denotes the real part of a complex number, and

$$\Psi(\boldsymbol{\tau}, v_2) = \left|\mathbf{I}_m + v_2^{-1}\tilde{\boldsymbol{\Sigma}}\mathbf{C}(\boldsymbol{\tau})\right|^{-\frac{1}{2}} \exp\left(-\frac{\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\boldsymbol{\mu}}}{2} + \frac{\boldsymbol{\xi}(\boldsymbol{\tau})'\left(\tilde{\boldsymbol{\Sigma}}^{-1} + v_2^{-1}\mathbf{C}(\boldsymbol{\tau})\right)^{-1}\boldsymbol{\xi}(\boldsymbol{\tau})}{2}\right). \tag{57}$$

The integral in (56) converges absolutely: uniformly in v_2,

$$|\Psi(\boldsymbol{\tau}, v_2)| \leq \exp\left(-\frac{\lambda_{\min}(\tilde{\boldsymbol{\Sigma}})\,\|\tilde{\mathbf{L}}\boldsymbol{\tau}\|^2}{2}\right), \tag{58}$$

where $\lambda_{\min}(\tilde{\boldsymbol{\Sigma}})$ denotes the smallest eigenvalue of $\tilde{\boldsymbol{\Sigma}}$.

Proof. From the proof of Theorem 2, conditional on $(\tilde{\mathbf{z}}, v_1, v_2)$, $\mathbf{q}_\mathbf{L}$ is normally distributed with mean $\boldsymbol{\eta} = \tilde{\mathbf{L}}'\tilde{\mathbf{z}}/v_1$ and covariance matrix $\boldsymbol{\Omega} = (\tilde{\mathbf{z}}'\tilde{\mathbf{z}})\,\tilde{\mathbf{L}}'\mathbf{Q}\mathbf{Q}'\tilde{\mathbf{L}}/(v_1^2 v_2)$, so its conditional characteristic function is, writing $\mathbf{w} = \tilde{\mathbf{L}}\mathbf{t}$,

$$\mathbb{E}\left[e^{\mathrm{i}\mathbf{t}'\mathbf{q}_\mathbf{L}} \,\middle|\, \tilde{\mathbf{z}}, v_1, v_2\right] = \exp\left(\frac{\mathrm{i}\,\mathbf{w}'\tilde{\mathbf{z}}}{v_1} - \frac{(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})(\mathbf{w}'\mathbf{w}) - (\mathbf{w}'\tilde{\mathbf{z}})^2}{2v_1^2 v_2}\right) = \exp\left(\tilde{\mathbf{z}}'\mathbf{M}\tilde{\mathbf{z}} + \mathrm{i}\,\mathbf{b}'\tilde{\mathbf{z}}\right), \tag{59}$$

where $\mathbf{M} = -\mathbf{C}(\mathbf{t})/(2v_1^2 v_2)$ and $\mathbf{b} = \mathbf{w}/v_1$, and we used $\mathbf{t}'\boldsymbol{\Omega}\mathbf{t} = [(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})(\mathbf{w}'\mathbf{w}) - (\mathbf{w}'\tilde{\mathbf{z}})^2]/(v_1^2 v_2)$. Note that this expression remains valid on the null event on which $\boldsymbol{\Omega}$ is singular. By the Cauchy–Schwarz inequality, $\mathbf{C}(\mathbf{t})$ is positive semidefinite, so $\tilde{\boldsymbol{\Sigma}}^{-1} - 2\mathbf{M} \succ 0$, and taking the expectation over $\tilde{\mathbf{z}} \sim \mathcal{N}_m(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}})$ by completing the square in the Gaussian density (see, e.g., Mathai and Provost, 1992, Chapter 3) gives

$$\begin{aligned}\mathbb{E}\left[e^{\mathrm{i}\mathbf{t}'\mathbf{q}_\mathbf{L}} \,\middle|\, v_1, v_2\right] &= \left|\mathbf{I}_m - 2\tilde{\boldsymbol{\Sigma}}\mathbf{M}\right|^{-\frac{1}{2}} \exp\left(-\frac{\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\boldsymbol{\mu}}}{2} + \frac{(\tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\boldsymbol{\mu}} + \mathrm{i}\mathbf{b})'(\tilde{\boldsymbol{\Sigma}}^{-1} - 2\mathbf{M})^{-1}(\tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\boldsymbol{\mu}} + \mathrm{i}\mathbf{b})}{2}\right) \\ &= \Psi\left(\frac{\mathbf{t}}{v_1}, v_2\right),\end{aligned} \tag{60}$$

where the last equality follows from the scaling property $\mathbf{C}(\mathbf{t}) = v_1^2\,\mathbf{C}(\mathbf{t}/v_1)$, so that $-2\mathbf{M} = v_2^{-1}\mathbf{C}(\mathbf{t}/v_1)$ and $\mathbf{b} = \tilde{\mathbf{L}}(\mathbf{t}/v_1)$.

We next establish (58). Since $\tilde{\boldsymbol{\Sigma}}\mathbf{C}(\boldsymbol{\tau})$ has the same eigenvalues as the positive semidefinite matrix $\tilde{\boldsymbol{\Sigma}}^{\frac{1}{2}}\mathbf{C}(\boldsymbol{\tau})\tilde{\boldsymbol{\Sigma}}^{\frac{1}{2}}$, the determinant factor in (57) is at most one. Writing $\mathbf{K} = (\tilde{\boldsymbol{\Sigma}}^{-1} + v_2^{-1}\mathbf{C}(\boldsymbol{\tau}))^{-1}$, $\mathbf{r} = \tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\boldsymbol{\mu}}$, and $\mathbf{w} = \tilde{\mathbf{L}}\boldsymbol{\tau}$, the real part of the exponent in (57) equals $-\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\boldsymbol{\mu}}/2 + (\mathbf{r}'\mathbf{K}\mathbf{r} - \mathbf{w}'\mathbf{K}\mathbf{w})/2$. Now $\mathbf{r}'\mathbf{K}\mathbf{r} \leq \mathbf{r}'\tilde{\boldsymbol{\Sigma}}\mathbf{r} = \tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\boldsymbol{\mu}}$ because $\mathbf{K} \preceq \tilde{\boldsymbol{\Sigma}}$, while $\mathbf{C}(\boldsymbol{\tau})\mathbf{w} = \mathbf{0}_m$ and the Cauchy–Schwarz inequality give

$$\mathbf{w}'\mathbf{K}\mathbf{w} \geq \frac{(\mathbf{w}'\mathbf{w})^2}{\mathbf{w}'(\tilde{\boldsymbol{\Sigma}}^{-1} + v_2^{-1}\mathbf{C}(\boldsymbol{\tau}))\mathbf{w}} = \frac{(\mathbf{w}'\mathbf{w})^2}{\mathbf{w}'\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{w}} \geq \lambda_{\min}(\tilde{\boldsymbol{\Sigma}})\,\mathbf{w}'\mathbf{w}, \tag{61}$$

which together yield (58).

Finally, the Gaussian bound (58) gives $|v_1^p\, e^{-\mathrm{i}v_1\boldsymbol{\tau}'\mathbf{q}_\mathbf{L}}\Psi(\boldsymbol{\tau}, v_2)| \leq v_1^p \exp(-\frac{1}{2}\lambda_{\min}(\tilde{\boldsymbol{\Sigma}})\|\tilde{\mathbf{L}}\boldsymbol{\tau}\|^2)$, whose product with $f_{n-m+1}(v_1)f_{n-m+2}(v_2)$ is integrable over $(\boldsymbol{\tau}, v_1, v_2)$ because $\mathbb{E}[v_1^p] < \infty$ and $\tilde{\mathbf{L}}$ has full column rank; this integrable dominating function justifies the interchange of expectation and integration below. Consequently, Fourier inversion together with Fubini's theorem gives

$$f_{\mathbf{q}_\mathbf{L}}(\mathbf{q}_\mathbf{L}) = \frac{1}{(2\pi)^p}\mathbb{E}_{v_1,v_2}\left[\int_{\mathbb{R}^p} e^{-\mathrm{i}\mathbf{t}'\mathbf{q}_\mathbf{L}}\,\Psi\left(\frac{\mathbf{t}}{v_1}, v_2\right)\,\mathrm{d}\mathbf{t}\right]$$

$$= \frac{1}{(2\pi)^p} \mathbb{E}_{v_2} \left[\int_{\mathbb{R}^p} \mathbb{E}_{v_1} \left[v_1^p \, e^{-\mathrm{i} v_1 \boldsymbol{\tau}' \mathbf{q}_\mathbf{L}} \right] \Psi(\boldsymbol{\tau}, v_2) \, \mathrm{d}\boldsymbol{\tau} \right], \tag{62}$$

where the inner substitution is $\mathbf{t} = v_1 \boldsymbol{\tau}$. Using the identity $v_1^p f_{n-m+1}(v_1) = 2^p \left((n-m+1)/2\right)_p f_{n-m+1+2p}(v_1)$ together with the characteristic function of the $\chi^2_{n-m+1+2p}$ distribution, we obtain

$$\mathbb{E}_{v_1} \left[v_1^p \, e^{-\mathrm{i} v_1 \theta} \right] = 2^p \left(\tfrac{n-m+1}{2} \right)_p (1 + 2\mathrm{i}\theta)^{-\frac{n-m+1+2p}{2}}, \tag{63}$$

and (56) follows, the real part being justified by the conjugate symmetry $\Psi(-\boldsymbol{\tau}, v_2) = \overline{\Psi(\boldsymbol{\tau}, v_2)}$. □

We single out the case $p = 1$, which is of primary practical interest. Setting $\mathbf{L} = \boldsymbol{\alpha}$ and writing $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}} \boldsymbol{\alpha}$, the vector $\boldsymbol{\tau}$ becomes a scalar τ, and folding the integral in (56) onto $(0, \infty)$ by the conjugate symmetry $\Psi(-\tau, v_2) = \overline{\Psi(\tau, v_2)}$ reduces it to the two-dimensional integral

$$f_{q_{\boldsymbol{\alpha}}}(q_{\boldsymbol{\alpha}}) = \frac{n-m+1}{\pi} \int_0^\infty \Re \left[(1 + 2\mathrm{i}\tau q_{\boldsymbol{\alpha}})^{-\frac{n-m+3}{2}} \, \bar{\Psi}(\tau) \right] \mathrm{d}\tau, \tag{64}$$

where $\bar{\Psi}(\tau) = \mathbb{E}_{v_2}[\Psi(\tau, v_2)]$ with $v_2 \sim \chi^2_{n-m+2}$, and $\Psi(\tau, v_2)$ is given by (57) with $\tilde{\mathbf{L}}$ replaced by the vector $\tilde{\boldsymbol{\alpha}}$; explicitly, $\mathbf{C}(\tau) = \tau^2[(\tilde{\boldsymbol{\alpha}}'\tilde{\boldsymbol{\alpha}})\mathbf{I}_m - \tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\alpha}}']$ and $\boldsymbol{\xi}(\tau) = \tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\boldsymbol{\mu}} + \mathrm{i}\tau\tilde{\boldsymbol{\alpha}}$. The outer integral over τ together with the inner expectation over v_2 makes (64) a two-dimensional quadrature, whose tail in τ is controlled by the Gaussian envelope (58).

Representation (64) is an attractive alternative to Theorem 2(iii) for the density of a single linear combination. By (33), the latter reduces to a two-dimensional integral only when $n - m$ is even; when $n - m$ is odd, the inner conditional expectation itself requires an additional integral (see Lemma 5), so that the density is then obtained from a three-dimensional integration problem. By contrast, (64) is always two-dimensional, irrespective of the parity of $n - m$, and is therefore generally faster to evaluate, the advantage being most pronounced when $n - m$ is odd.

For $p = 2$ and $p = 3$, Proposition 1 yields three- and four-dimensional integrals that are well within reach of standard cubature routines, with the Gaussian envelope (58) providing explicit truncation bounds. Combined with the marginalization approach described at the end of this section, which requires an $(m - p + 1)$-dimensional integral, and with the six-dimensional route based on the joint density of $(u, v, \hat{u})$, the density of $\mathbf{q}_\mathbf{L}$ can always be computed from an integral of dimension at most $\min(p + 1, m - p + 1, 6)$. The six-dimensional bound, being free of p and m, becomes the binding one only when $p > 5$ and $m - p > 5$. Dimension is not the only consideration, however: the integrand in (56) is smooth with the explicit Gaussian envelope (58), whereas the six-dimensional route embeds an oscillatory two-dimensional inversion integral at every quadrature node. In the regime where the six-dimensional bound binds, deterministic cubature is demanding in any case, and simulation-based evaluation of (29), which draws $(u, v, \hat{u}, v_1)$ directly and averages the closed-form kernel, is an attractive alternative whose accuracy does not deteriorate with p or m.

The density at the origin admits a simpler form. For $p < m$ and $\mathbf{q}_\mathbf{L} = \mathbf{0}_p$, we have $\theta_1 = \theta_2 = 0$ and $a = u/\hat{u}$, so

$$f_{\mathbf{q}_\mathbf{L}}(\mathbf{0}_p) = \frac{2^p \, \Gamma\left(\frac{n-m+2+p}{2}\right) \left(\frac{n-m+1}{2}\right)_p}{\Gamma\left(\frac{n-m+2}{2}\right) \pi^{\frac{p}{2}} |\tilde{\mathbf{L}}'\tilde{\mathbf{L}}|^{\frac{1}{2}}} \, \mathbb{E} \left[\frac{\hat{u}^{\frac{n-m+1+p}{2}}}{(u + \hat{u})^{\frac{n-m+1}{2} + p}} \right], \tag{65}$$

and, in particular, for $p = 1$,

$$f_{q_{\boldsymbol{\alpha}}}(0) = \frac{(n-m+1) \, \Gamma\left(\frac{n-m+3}{2}\right)}{\sqrt{\pi} \, \Gamma\left(\frac{n-m+2}{2}\right) (\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w^{-1}\boldsymbol{\alpha})^{\frac{1}{2}}} \, \mathbb{E} \left[\frac{\hat{u}^{\frac{n-m+2}{2}}}{(\hat{z}^2 + \hat{u})^{\frac{n-m+3}{2}}} \right]. \tag{66}$$

These unconditional expectations are moments of ratios of quadratic forms in normal random variables and can be computed by the methods of Bao and Kan (2013) and Hillier et al. (2014).

When p is close to m, an alternative to case (ii) is available. Let $\mathbf{Q}_1$ be an $m \times (m-p)$ orthonormal matrix whose columns are orthogonal to $\mathbf{L}$, set $\mathbf{Q} = [\mathbf{L},\ \mathbf{Q}_1]$, and let $\tilde{\mathbf{h}} = \mathbf{Q}'\mathbf{W}^{-1}\mathbf{z}$, whose first p elements equal $\mathbf{q}_\mathbf{L} = \mathbf{L}'\mathbf{W}^{-1}\mathbf{z}$. By case (i), the density of $\tilde{\mathbf{h}}$ is

$$f_{\tilde{\mathbf{h}}}(\tilde{\mathbf{h}}) = |\mathbf{Q}'\mathbf{Q}|^{-\frac{1}{2}} f_\mathbf{q}\Big((\mathbf{Q}^{-1})'\tilde{\mathbf{h}}\Big), \tag{67}$$

so that, partitioning $\tilde{\mathbf{h}} = [\tilde{\mathbf{h}}_1',\ \tilde{\mathbf{h}}_2']'$, the density of $\mathbf{q}_\mathbf{L} = \tilde{\mathbf{h}}_1$ is obtained by integrating out $\tilde{\mathbf{h}}_2$:

$$f_{\mathbf{q}_\mathbf{L}}(\mathbf{q}_\mathbf{L}) = \int_{\mathbb{R}^{m-p}} f_{\tilde{\mathbf{h}}}(\mathbf{q}_\mathbf{L}, \tilde{\mathbf{h}}_2)\, \mathrm{d}\tilde{\mathbf{h}}_2. \tag{68}$$

Since $f_{\tilde{\mathbf{h}}}(\cdot)$ is itself available as a single integral, this approach is attractive when $m-p$ is small, in particular for $p = m-1$ or $p = m-2$; for $p < m-2$, the dimension of the outer integral renders it impractical.

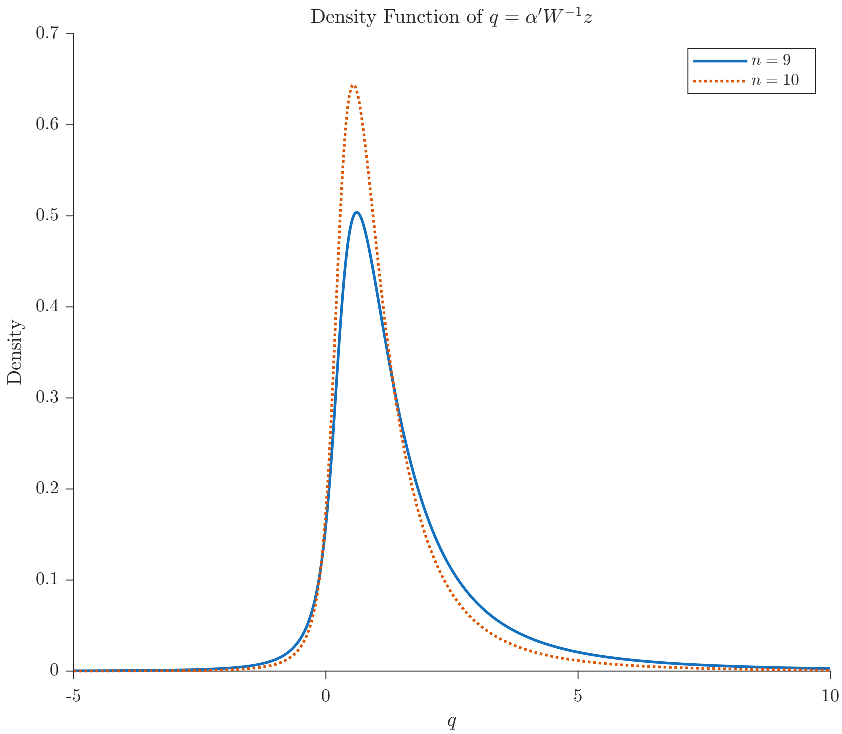

Figure 1: Density function of $q_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{W}^{-1}\mathbf{z}$ for $n = 9$ (solid line) and $n = 10$ (dotted line), where $m = 5$, $\boldsymbol{\alpha} = \mathbf{1}_5$, $\boldsymbol{\mu} = [1,\ 2,\ 3,\ 4,\ 5]'$, $\boldsymbol{\Sigma}_w = 0.5\,\mathbf{I}_5 + 0.5\,\mathbf{1}_5\mathbf{1}_5'$, and $\boldsymbol{\Sigma}_z = 0.2\,\mathbf{I}_5 + 0.8\,\mathbf{1}_5\mathbf{1}_5'$. The density is evaluated by Gauss–Legendre quadrature of the representation in Theorem 2(iii); the two curves are distinguished by line style rather than colour.

Figure 1 illustrates Theorem 2(iii) by plotting the density of $q_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{W}^{-1}\mathbf{z}$ for $m = 5$ and $n \in \{9, 10\}$, with the parameters given in the caption. Both densities are unimodal and right-skewed, with modes that lie close to each other. The smaller value of n places less probability mass around the mode and produces a heavier right tail, reflecting the greater variability of $\mathbf{W}^{-1}$ when the degrees of freedom are low.

4 Proportional covariance matrices

This section presents the simplifications that arise in the important special case of proportional covariance matrices, $\boldsymbol{\Sigma}_w = \boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}_z = \lambda\boldsymbol{\Sigma}$ for some $\lambda > 0$. This case covers, among other settings, the situation in which the roles of $\mathbf{W}$ and $\mathbf{z}$ are played by the sample covariance matrix and the sample mean vector computed from a normal random sample, as in (1). Taking $\mathbf{W} = \hat{\boldsymbol{\Sigma}} \sim \mathcal{W}_m(N-1, \boldsymbol{\Sigma}/(N-1))$ and $\mathbf{z} = \hat{\boldsymbol{\mu}} \sim \mathcal{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}/N)$ gives $\boldsymbol{\Sigma}_w = \boldsymbol{\Sigma}/(N-1)$ and $\boldsymbol{\Sigma}_z = \boldsymbol{\Sigma}/N = \frac{N-1}{N}\boldsymbol{\Sigma}_w$, so that the proportionality constant is $\lambda = (N-1)/N$, and $\mathbf{W}^{-1}\mathbf{z} = \hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}$ is the (unnormalized) plug-in mean–variance (or discriminant) vector.

The proportional setting is precisely the framework of Bodnar and Okhrin (2011), whose analysis of the exact distribution centres on a single linear combination $\boldsymbol{\alpha}'\mathbf{W}^{-1}\mathbf{z}$, so it is worth spelling out what the results below add to theirs. Theorem 3(iv) recovers the single-integral representation of the density of $\boldsymbol{\alpha}'\mathbf{W}^{-1}\mathbf{z}$ in terms of the parabolic cylinder function, in the spirit of Bodnar and Okhrin (2011). The remaining results, however, have no counterpart in their paper. First, Corollary 1 provides the joint density of the *entire* vector $\mathbf{W}^{-1}\mathbf{z}$ as a one-dimensional integral. Second, parts (i)–(iii) of Theorem 3 deliver the joint density of an arbitrary number $p \le m$ of linear combinations $\mathbf{L}'\mathbf{W}^{-1}\mathbf{z}$: a one-dimensional integral for $p = m$, a double integral for $p = m-1$, and a triple integral for general $p < m-1$, the last of which, as shown in Remark 1 below, collapses to a double integral whenever $m-p$ is odd. Third, Remark 2 complements part (iv) by supplying explicit finite-sum expressions, in terms of confluent hypergeometric functions, for the conditional expectation that drives the density of a single linear combination whenever $n-m$ is even. Finally, all results accommodate an arbitrary proportionality constant $\lambda > 0$ and real-valued degrees of freedom $n > m-1$. In short, Theorem 3 shows that the representations of Theorem 2 simplify considerably in the proportional setting: low-dimensional integral formulas become available for every value of p, not only for $p = 1$.

We begin with the density of the entire vector $\mathbf{W}^{-1}\mathbf{z}$, which corresponds to $\mathbf{L} = \mathbf{I}_m$. When $\boldsymbol{\Sigma}_w = \boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}_z = \lambda\boldsymbol{\Sigma}$, we have $\tilde{\boldsymbol{\Sigma}} = \lambda\mathbf{I}_m$, so that $C_{11} = 1/\lambda$, $\mathbf{C}_{21} = \mathbf{0}_{m-1}$, and $\mathbf{C}_{22} = \mathbf{I}_{m-1}/\lambda$, and Theorem 1 simplifies as follows.

Corollary 1. *Under the assumptions of Theorem 1, with $\boldsymbol{\Sigma}_w = \boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}_z = \lambda\boldsymbol{\Sigma}$ for some $\lambda > 0$, the density function of $\mathbf{q}$ is given by*

$$f_{\mathbf{q}}(\mathbf{q}) = \frac{\exp\left(-\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{2\lambda}\right)|\boldsymbol{\Sigma}|^{\frac{1}{2}}(n-m+1)_m}{(2\pi\lambda)^{\frac{m}{2}}} \times \int_0^\infty \left[1 + \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})u_1}{\lambda}\right]^{-\frac{m-1}{2}} \exp\left(b_0 + \frac{c u_1}{2\left[1 + \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})u_1}{\lambda}\right]}\right) f_{n+2}(u_1)\,du_1, \tag{69}$$

where $\hat{\mathbf{q}} = \boldsymbol{\Sigma}^{\frac{1}{2}}\mathbf{q}$, $\hat{\boldsymbol{\mu}} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\mu}/\lambda$, and

$$b_0 = (\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}})u_1 - \frac{(\hat{\mathbf{q}}'\hat{\mathbf{q}})u_1^2}{2\lambda}, \qquad c = (\hat{\mathbf{q}}'\hat{\mathbf{q}})(\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}) - (\hat{\mathbf{q}}'\hat{\boldsymbol{\mu}})^2. \tag{70}$$

Unlike in Theorem 1, no separate treatment of the origin is needed in Corollary 1: since $\hat{\mathbf{q}} = \boldsymbol{\Sigma}^{\frac{1}{2}}\mathbf{q}$ enters the formula without normalization, the representation is well defined at $\mathbf{q} = \mathbf{0}_m$, where $b_0 = c = 0$ and the integral equals one, so that it reduces to

$$f_{\mathbf{q}}(\mathbf{0}_m) = \frac{\exp\left(-\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{2\lambda}\right)|\boldsymbol{\Sigma}|^{\frac{1}{2}}(n-m+1)_m}{(2\pi\lambda)^{\frac{m}{2}}}, \tag{71}$$

in agreement with (10).

Theorem 3. *Let $\mathbf{W} \sim \mathcal{W}_m(n, \boldsymbol{\Sigma})$ and $\mathbf{z} \sim \mathcal{N}_m(\boldsymbol{\mu}, \lambda\boldsymbol{\Sigma})$ be independent, with $m \ge 2$, $n > m-1$, $\lambda > 0$, and positive definite $\boldsymbol{\Sigma}$. Let $\mathbf{L}$ be an $m \times p$ matrix of constants with rank $p \le m$, and let $\mathbf{Q} = [\tilde{\mathbf{L}},\ \mathbf{Q}_1]$ be an $m \times m$ orthonormal matrix, where*

$$\tilde{\mathbf{L}} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{L}(\mathbf{L}'\boldsymbol{\Sigma}^{-1}\mathbf{L})^{-\frac{1}{2}} \tag{72}$$

and $\mathbf{Q}_1$ *is an* $m \times (m-p)$ *orthonormal matrix whose columns are orthogonal to* $\tilde{\mathbf{L}}$. *Define*

$$\hat{\mathbf{W}} = \mathbf{Q}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{W}\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{Q} \sim \mathcal{W}_m(n, \mathbf{I}_m), \qquad \hat{\mathbf{z}} = \mathbf{Q}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{z}/\sqrt{\lambda} \sim \mathcal{N}_m(\hat{\boldsymbol{\mu}}, \mathbf{I}_m), \tag{73}$$

with $\hat{\boldsymbol{\mu}} = \mathbf{Q}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\mu}/\sqrt{\lambda}$, *and partition* $\hat{\mathbf{q}} = \hat{\mathbf{W}}^{-1}\hat{\mathbf{z}} = [\hat{\mathbf{q}}_1', \ \hat{\mathbf{q}}_2']'$ *and* $\hat{\boldsymbol{\mu}} = [\hat{\boldsymbol{\mu}}_1', \ \hat{\boldsymbol{\mu}}_2']'$, *where* $\hat{\mathbf{q}}_1$ *and* $\hat{\boldsymbol{\mu}}_1$ *contain the first* p *elements of* $\hat{\mathbf{q}}$ *and* $\hat{\boldsymbol{\mu}}$, *respectively. Then the density function of* $\mathbf{q}_{\mathbf{L}} = \mathbf{L}'\mathbf{W}^{-1}\mathbf{z}$ *is given as follows.*

(i) For $p = m$:

$$f_{\mathbf{q}_{\mathbf{L}}}(\mathbf{q}_{\mathbf{L}}) = |\mathbf{L}'\mathbf{L}|^{-\frac{1}{2}} f_{\mathbf{q}}\left((\mathbf{L}^{-1})'\mathbf{q}_{\mathbf{L}}\right), \tag{74}$$

where $f_{\mathbf{q}}(\cdot)$ *is given in Corollary 1.*

(ii) For $p = m-1$:

$$f_{\mathbf{q}_{\mathbf{L}}}(\mathbf{q}_{\mathbf{L}}) = \frac{\Gamma(n)\exp\left(-\frac{\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}}{2}\right)|\lambda\mathbf{L}'\boldsymbol{\Sigma}^{-1}\mathbf{L}|^{-\frac{1}{2}}}{\Gamma(n-m+1)(2\pi)^{\frac{m}{2}}} \int_0^\infty \int_{-\infty}^\infty \frac{u}{[1+(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+\hat{q}_2^2)u]^{\frac{m-1}{2}}}$$
$$\times \exp\left((\hat{\mathbf{q}}_1'\hat{\boldsymbol{\mu}}_1 + \hat{q}_2\hat{\mu}_2)u - \frac{(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+\hat{q}_2^2)u^2}{2} + \frac{uc}{2[1+(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+\hat{q}_2^2)u]}\right) f_n(u)\,\mathrm{d}\hat{q}_2\,\mathrm{d}u, \tag{75}$$

where $c = (\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+\hat{q}_2^2)(\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}) - (\hat{\mathbf{q}}_1'\hat{\boldsymbol{\mu}}_1+\hat{q}_2\hat{\mu}_2)^2$.

(iii) For $p < m-1$:

$$f_{\mathbf{q}_{\mathbf{L}}}(\mathbf{q}_{\mathbf{L}}) = \frac{\Gamma(n)\exp\left(-\frac{\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}}{2}\right)|\lambda\mathbf{L}'\boldsymbol{\Sigma}^{-1}\mathbf{L}|^{-\frac{1}{2}}}{\Gamma(n-m+1)\Gamma\left(\frac{m-p-1}{2}\right)2^{\frac{m}{2}}\pi^{\frac{p+1}{2}}}$$
$$\times \int_0^\infty \int_0^\infty \int_{-1}^1 \frac{uv^{\frac{1}{2}}[v(1-x^2)]^{\frac{m-p-3}{2}}}{[1+(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+v)u]^{\frac{m-1}{2}}}$$
$$\times \exp\left([\hat{\mathbf{q}}_1'\hat{\boldsymbol{\mu}}_1 + (\hat{\boldsymbol{\mu}}_2'\hat{\boldsymbol{\mu}}_2)^{\frac{1}{2}}\sqrt{v}\,x]u - \frac{(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+v)u^2}{2} + \frac{uc}{2+2(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+v)u}\right)$$
$$\times f_n(u)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}u, \tag{76}$$

where $c = (\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+v)(\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}) - [\hat{\mathbf{q}}_1'\hat{\boldsymbol{\mu}}_1 + (\hat{\boldsymbol{\mu}}_2'\hat{\boldsymbol{\mu}}_2)^{\frac{1}{2}}\sqrt{v}\,x]^2$.

(iv) For $p = 1$ *with* $\mathbf{L} = \boldsymbol{\alpha}$ *and* $q_{\boldsymbol{\alpha}} \neq 0$:

$$f_{q_{\boldsymbol{\alpha}}}(q_{\boldsymbol{\alpha}}) = \frac{1}{\sqrt{\lambda\,\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}} \int_0^\infty p(\hat{q}_1, y)\, g^{\delta}_{m-1,n-m+2}(y)\,\mathrm{d}y, \tag{77}$$

where $\hat{q}_1 = q_{\boldsymbol{\alpha}}/\sqrt{\lambda\,\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}$,

$$\delta = \frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}}{\lambda}, \tag{78}$$

$$p(\hat{q}_1, y) = \frac{(n-m+1)\,\phi(\hat{\mu}_1/\sqrt{1+y})\,(1+y)^{\frac{n-m+1}{4}} e^{\frac{x^2}{4}}}{(4\hat{q}_1^2)^{\frac{n-m+3}{4}}} D_{-\frac{n-m+3}{2}}(x), \tag{79}$$

and

$$x = \frac{(1+y) - 2\hat{\mu}_1\hat{q}_1}{2\sqrt{1+y}\,|\hat{q}_1|}; \tag{80}$$

here $\phi(\cdot)$ denotes the standard normal density function, $g^{\delta}_{m,n}(\cdot)$ the density function of the ratio $\chi^2_m(\delta)/\chi^2_n$ of independent chi-square random variables, and $D_\nu(\cdot)$ the parabolic cylinder function. The density is finite at $q_{\boldsymbol{\alpha}} = 0$; setting $\hat{q}_1 = 0$ in the mixture representation (94) obtained in the proof below gives

$$f_{q_{\boldsymbol{\alpha}}}(0) = \frac{1}{\sqrt{\lambda\,\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}} \int_0^\infty \frac{n-m+1}{\sqrt{2\pi(1+y)}} \exp\left(-\frac{\hat{\mu}_1^2}{2(1+y)}\right) g^{\delta}_{m-1,n-m+2}(y)\,\mathrm{d}y. \tag{81}$$

This mixture form is also a numerically stable representation in a neighbourhood of $q_{\boldsymbol{\alpha}} = 0$, where the prefactor $(4\hat{q}_1^2)^{-(n-m+3)/4}$ and $D_{-(n-m+3)/2}(x)$ diverge individually while their product stays finite.

Proof. Part (i) follows as in the proof of Theorem 2(i).

Suppose now $p < m$. Let $\tilde{\mathbf{W}} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{W}\boldsymbol{\Sigma}^{-\frac{1}{2}}$ and $\tilde{\mathbf{z}} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{z}/\sqrt{\lambda}$. Using the definitions of $\mathbf{Q}$, $\hat{\mathbf{W}} = \mathbf{Q}'\tilde{\mathbf{W}}\mathbf{Q}$, and $\hat{\mathbf{z}} = \mathbf{Q}'\tilde{\mathbf{z}}$, we can write

$$\mathbf{q}_{\mathbf{L}} = \mathbf{L}'\mathbf{W}^{-1}\mathbf{z} = \sqrt{\lambda}\,\mathbf{L}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{Q}\hat{\mathbf{W}}^{-1}\hat{\mathbf{z}}. \tag{82}$$

Since

$$\sqrt{\lambda}\,\mathbf{L}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{Q} = [\sqrt{\lambda}(\mathbf{L}'\boldsymbol{\Sigma}^{-1}\mathbf{L})^{\frac{1}{2}},\ \mathbf{0}_{p\times(m-p)}], \tag{83}$$

it follows that

$$\mathbf{q}_{\mathbf{L}} = \sqrt{\lambda}\,(\mathbf{L}'\boldsymbol{\Sigma}^{-1}\mathbf{L})^{\frac{1}{2}}\hat{\mathbf{q}}_1, \tag{84}$$

where $\hat{\mathbf{q}}_1$ collects the first p elements of $\hat{\mathbf{q}} = \hat{\mathbf{W}}^{-1}\hat{\mathbf{z}}$, and hence

$$f_{\mathbf{q}_{\mathbf{L}}}(\mathbf{q}_{\mathbf{L}}) = |\lambda\,\mathbf{L}'\boldsymbol{\Sigma}^{-1}\mathbf{L}|^{-\frac{1}{2}} f_{n,\hat{\boldsymbol{\mu}}}(\hat{\mathbf{q}}_1), \tag{85}$$

where $f_{n,\hat{\boldsymbol{\mu}}}(\cdot)$ denotes the density of the relevant subvector of $\hat{\mathbf{W}}^{-1}\hat{\mathbf{z}}$ with $\hat{\mathbf{W}} \sim \mathcal{W}_m(n, \mathbf{I}_m)$ and $\hat{\mathbf{z}} \sim \mathcal{N}_m(\hat{\boldsymbol{\mu}}, \mathbf{I}_m)$. It remains to derive $f_{n,\hat{\boldsymbol{\mu}}}(\hat{\mathbf{q}}_1)$, the marginal density of the first p elements. By Corollary 1 (with $\boldsymbol{\Sigma} = \mathbf{I}_m$ and $\lambda = 1$), the joint density of $\hat{\mathbf{q}}$ is available as a single integral, and the marginal is obtained by integrating out $\hat{\mathbf{q}}_2$.

When $p = m-1$, $\hat{\mathbf{q}}_2$ is a scalar and direct integration gives

$$\begin{aligned}
f_{n,\hat{\boldsymbol{\mu}}}(\hat{\mathbf{q}}_1) &= \int_{-\infty}^{\infty} f_{n,\hat{\boldsymbol{\mu}}}(\hat{\mathbf{q}}_1, \hat{q}_2)\,\mathrm{d}\hat{q}_2 \\
&= \frac{\Gamma(n)\exp\left(-\frac{\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}}{2}\right)}{\Gamma(n-m+1)(2\pi)^{\frac{m}{2}}} \int_0^\infty \int_{-\infty}^{\infty} \frac{u}{[1+(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1 + \hat{q}_2^2)u]^{\frac{m-1}{2}}} \\
&\quad \times \exp\left((\hat{\mathbf{q}}_1'\hat{\boldsymbol{\mu}}_1 + \hat{q}_2\hat{\mu}_2)u - \frac{(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1 + \hat{q}_2^2)u^2}{2} + \frac{uc}{2[1+(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1 + \hat{q}_2^2)u]}\right) f_n(u)\,\mathrm{d}\hat{q}_2\,\mathrm{d}u,
\end{aligned} \tag{86}$$

with c as given in the statement, which proves part (ii).

When $p < m-1$, we start from the analogous $(m-p)$-dimensional integral

$$f_{n,\hat{\boldsymbol{\mu}}}(\hat{\mathbf{q}}_1) = \int_{\mathbb{R}^{m-p}} f_{n,\hat{\boldsymbol{\mu}}}(\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2)\,\mathrm{d}\hat{\mathbf{q}}_2$$

$$
\begin{aligned}
&= \frac{\Gamma(n)\exp\left(-\frac{\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}}{2}\right)}{\Gamma(n-m+1)(2\pi)^{\frac{m}{2}}} \int_0^\infty \int_{\mathbb{R}^{m-p}} \frac{u}{[1+(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+\hat{\mathbf{q}}_2'\hat{\mathbf{q}}_2)u]^{\frac{m-1}{2}}} \\
&\quad \times \exp\left((\hat{\mathbf{q}}_1'\hat{\boldsymbol{\mu}}_1+\hat{\mathbf{q}}_2'\hat{\boldsymbol{\mu}}_2)u - \frac{(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+\hat{\mathbf{q}}_2'\hat{\mathbf{q}}_2)u^2}{2} + \frac{uc}{2[1+(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+\hat{\mathbf{q}}_2'\hat{\mathbf{q}}_2)u]}\right) \\
&\quad \times f_n(u)\,\mathrm{d}\hat{\mathbf{q}}_2\,\mathrm{d}u, \qquad (87)
\end{aligned}
$$

where $c = (\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+\hat{\mathbf{q}}_2'\hat{\mathbf{q}}_2)(\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}) - (\hat{\mathbf{q}}_1'\hat{\boldsymbol{\mu}}_1+\hat{\mathbf{q}}_2'\hat{\boldsymbol{\mu}}_2)^2$. The integrand depends on $\hat{\mathbf{q}}_2$ only through the linear form $\hat{\boldsymbol{\mu}}_2'\hat{\mathbf{q}}_2$ and the quadratic form $\hat{\mathbf{q}}_2'\hat{\mathbf{q}}_2$. By Fang and Zhang (1990, Lemma 3.4.1), for every nonnegative Borel function g,

$$
\int_{\mathbb{R}^{m-p}} g(\boldsymbol{\alpha}'\mathbf{x},\mathbf{x}'\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{\pi^{\frac{m-p-1}{2}}}{\Gamma\left(\frac{m-p-1}{2}\right)} \int_0^\infty \int_{-\infty}^\infty w^{\frac{m-p-3}{2}} g\left((\boldsymbol{\alpha}'\boldsymbol{\alpha})^{\frac{1}{2}}y,\, y^2+w\right)\,\mathrm{d}y\,\mathrm{d}w, \qquad (88)
$$

and the change of variables $v = w + y^2$, $x = y/\sqrt{v}$ gives

$$
\int_{\mathbb{R}^{m-p}} g(\boldsymbol{\alpha}'\mathbf{x},\mathbf{x}'\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{\pi^{\frac{m-p-1}{2}}}{\Gamma\left(\frac{m-p-1}{2}\right)} \int_0^\infty \int_{-1}^1 [v(1-x^2)]^{\frac{m-p-3}{2}} g\left((\boldsymbol{\alpha}'\boldsymbol{\alpha})^{\frac{1}{2}}\sqrt{v}\,x,\, v\right)\sqrt{v}\,\mathrm{d}x\,\mathrm{d}v. \qquad (89)
$$

Applying (89) yields the triple-integral representation in part (iii).

For part (iv), we return to the identity-scale representation $q_{\boldsymbol{\alpha}} = \sqrt{\lambda\,\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}\,\hat{q}_1$ from (84), where now $\hat{q}_1 = \mathbf{e}_1'\hat{\mathbf{W}}^{-1}\hat{\mathbf{z}}$ is the first element of $\hat{\mathbf{W}}^{-1}\hat{\mathbf{z}}$, with $\hat{\mathbf{W}} \sim \mathcal{W}_m(n,\mathbf{I}_m)$ and $\hat{\mathbf{z}} \sim \mathcal{N}_m(\hat{\boldsymbol{\mu}},\mathbf{I}_m)$ independent and $\hat{\boldsymbol{\mu}} = [\hat{\mu}_1,\ \hat{\boldsymbol{\mu}}_2']'$. Let $\hat{\mathbf{s}} = \hat{\mathbf{z}}/(\hat{\mathbf{z}}'\hat{\mathbf{z}})^{\frac{1}{2}}$ and let $\mathbf{Q}_1$ be an $m\times(m-1)$ orthonormal matrix whose columns are orthogonal to $\hat{\mathbf{z}}$. Writing $\mathbf{e}_1 = (\mathbf{e}_1'\hat{\mathbf{s}})\hat{\mathbf{s}} + \mathbf{Q}_1\mathbf{Q}_1'\mathbf{e}_1$ and applying Lemma 3 with $\mathbf{q} = \hat{\mathbf{z}}$ (conditionally on $\hat{\mathbf{z}}$, which is independent of $\hat{\mathbf{W}}$),

$$
\hat{q}_1 = (\hat{\mathbf{z}}'\hat{\mathbf{z}})^{\frac{1}{2}}\left[(\mathbf{e}_1'\hat{\mathbf{s}})\,\hat{\mathbf{s}}'\hat{\mathbf{W}}^{-1}\hat{\mathbf{s}} + (\mathbf{Q}_1'\mathbf{e}_1)'\mathbf{Q}_1'\hat{\mathbf{W}}^{-1}\hat{\mathbf{s}}\right] \overset{d}{=} \frac{1}{v_1}\left(\hat{z}_1 + \frac{\zeta}{\sqrt{v_2}}\right), \qquad (90)
$$

where $v_1 \sim \chi^2_{n-m+1}$, $v_2 \sim \chi^2_{n-m+2}$, $\hat{z}_1 = \mathbf{e}_1'\hat{\mathbf{z}}$ is the first element of $\hat{\mathbf{z}}$, and, conditionally on $\hat{\mathbf{z}}$, $\zeta = (\hat{\mathbf{z}}'\hat{\mathbf{z}})^{\frac{1}{2}}(\mathbf{Q}_1'\mathbf{e}_1)'\mathbf{x} \sim \mathcal{N}(0,\hat{u})$ with $\mathbf{x} \sim \mathcal{N}_{m-1}(\mathbf{0}_{m-1},\mathbf{I}_{m-1})$ and $\hat{u} = \hat{\mathbf{z}}'\hat{\mathbf{z}} - \hat{z}_1^2 = \sum_{i=2}^m \hat{z}_i^2$; here v_1, v_2, and $\mathbf{x}$ are mutually independent and independent of $\hat{\mathbf{z}}$. Since $\hat{z}_1 \sim \mathcal{N}(\hat{\mu}_1,1)$ and $\hat{u} \sim \chi^2_{m-1}(\delta)$, with

$$
\delta = \hat{\boldsymbol{\mu}}_2'\hat{\boldsymbol{\mu}}_2 = \frac{1}{\lambda}\left[\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}\right], \qquad (91)
$$

are independent, it follows that, conditional on $(\hat{u},v_1,v_2)$,

$$
\hat{q}_1\,|\,\hat{u},v_1,v_2 \sim \mathcal{N}\left(\frac{\hat{\mu}_1}{v_1},\ \frac{1+\hat{u}/v_2}{v_1^2}\right). \qquad (92)
$$

Writing $y = \hat{u}/v_2$, which has the density $g^{\delta}_{m-1,n-m+2}$ of the ratio $\chi^2_{m-1}(\delta)/\chi^2_{n-m+2}$ and is independent of v_1, and integrating out $(\hat{u},v_2)$ through y, we obtain

$$
f_{n,\hat{\boldsymbol{\mu}}}(\hat{q}_1) = \int_0^\infty p(\hat{q}_1,y)\,g^{\delta}_{m-1,n-m+2}(y)\,\mathrm{d}y, \qquad (93)
$$

where, with $v_1 \sim \chi^2_{n-m+1}$,

$$
p(\hat{q}_1,y) = \mathbb{E}_{v_1}\left[\frac{v_1}{\sqrt{2\pi(1+y)}}\exp\left(-\frac{(\hat{q}_1v_1-\hat{\mu}_1)^2}{2(1+y)}\right)\right]. \qquad (94)
$$

Expanding the χ^2_{n-m+1} density and using the integral representation of the parabolic cylinder function,

$$\int_0^\infty t^{\kappa-1} e^{-\beta t^2-\gamma t}\,\mathrm{d}t = (2\beta)^{-\frac{\kappa}{2}}\Gamma(\kappa)\, e^{\frac{\gamma^2}{8\beta}} D_{-\kappa}\left(\frac{\gamma}{\sqrt{2\beta}}\right), \qquad \beta > 0,\ \kappa > 0, \tag{95}$$

with $\kappa = (n-m+3)/2$, $\beta = \hat{q}_1^2/[2(1+y)]$, and $\gamma = [(1+y) - 2\hat{\mu}_1\hat{q}_1]/[2(1+y)]$, evaluates the expectation and yields $p(\hat{q}_1, y)$ as stated in part (iv), with $x = \gamma/\sqrt{2\beta}$. Finally, $f_{q_{\boldsymbol{\alpha}}}(q_{\boldsymbol{\alpha}}) = |\lambda\, \boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}|^{-\frac{1}{2}} f_{n,\hat{\boldsymbol{\mu}}}(\hat{q}_1)$ completes part (iv); for $\lambda = 1$ this recovers the result of Bodnar and Okhrin (2011). This also shows that the case $p = 1$, already contained in part (iii) for $m \geq 3$, admits the more convenient single-integral form given in part (iv). □

Remark 1. *Assume first that $\hat{\boldsymbol{\mu}}_2 \neq \mathbf{0}_{m-p}$, equivalently $\delta > 0$. When $m-p$ is odd—equivalently, $\nu \equiv (m-p-3)/2$ is a nonnegative integer—the triple integral in part (iii) of Theorem 3 collapses to a double integral, because the inner integral over x in (76) can be evaluated in closed form. Collecting the terms of the exponent in (76) that depend on x, we need, for $a > 0$,*

$$\int_{-1}^1 e^{-ax^2+bx}(1-x^2)^\nu\,\mathrm{d}x = \frac{e^{\frac{b^2}{4a}}}{a^{\nu+\frac{1}{2}}}\int_{x_1}^{x_2} e^{-t^2}(x_2-t)^\nu(t-x_1)^\nu\,\mathrm{d}t, \tag{96}$$

obtained via the transformation $t = \sqrt{a}\,x - b/(2\sqrt{a})$, where

$$\begin{aligned} a &= \frac{\hat{\boldsymbol{\mu}}_2'\hat{\boldsymbol{\mu}}_2\, vu}{2[1+(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+v)u]}, \qquad b = \sqrt{\hat{\boldsymbol{\mu}}_2'\hat{\boldsymbol{\mu}}_2\, v}\; u\left(1 - \frac{\hat{\mathbf{q}}_1'\hat{\boldsymbol{\mu}}_1}{1+(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+v)u}\right), \\ x_1 &= -\frac{2a+b}{2\sqrt{a}}, \qquad x_2 = \frac{2a-b}{2\sqrt{a}}. \end{aligned} \tag{97}$$

Introduce

$$h_k(x) = \int_{-\infty}^x e^{-t^2}(x-t)^k\,\mathrm{d}t = \frac{D_{-k-1}(-\sqrt{2}x)\,k!\,e^{-\frac{x^2}{2}}}{2^{\frac{k+1}{2}}}, \tag{98}$$

where $D_{-k-1}(z) = e^{-\frac{z^2}{4}}\Gamma(k+1)^{-1}\int_0^\infty t^k e^{-\frac{t^2}{2}-tz}\,\mathrm{d}t$ is the parabolic cylinder function; the h_k obey $h_0(x) = \frac{\sqrt{\pi}}{2}\operatorname{erfc}(-x)$, $h_1(x) = \frac{1}{2}[e^{-x^2} + \sqrt{\pi}\,x\operatorname{erfc}(-x)]$, and $h_k(x) = xh_{k-1}(x) + \frac{k-1}{2}h_{k-2}(x)$ for $k > 1$, so they are readily evaluated. Using $x_2 - x_1 = 2\sqrt{a}$ and a binomial expansion,

$$\int_{-1}^1 e^{-ax^2+bx}(1-x^2)^\nu\,\mathrm{d}x = \frac{e^{\frac{b^2}{4a}}(-1)^\nu 2^\nu}{a^{\frac{\nu+1}{2}}}\sum_{k=0}^{\nu}\binom{\nu}{k}\frac{(-1)^{\nu+k}h_{\nu+k}(x_2) - h_{\nu+k}(x_1)}{(2\sqrt{a})^k}. \tag{99}$$

Substituting this identity into (76) yields the double integral

$$\begin{aligned} f_{\mathbf{q}_{\mathbf{L}}}(\mathbf{q}_{\mathbf{L}}) = {} & \frac{\Gamma(n)\,|\lambda\mathbf{L}'\boldsymbol{\Sigma}^{-1}\mathbf{L}|^{-\frac{1}{2}}}{\Gamma(n-m+1)2^{\frac{p+3}{2}}\pi^{\frac{p+1}{2}}\,\Gamma(\nu+1)} \\ & \times \int_0^\infty\int_0^\infty \frac{uv^{\nu+\frac{1}{2}}}{[1+(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+v)u]^{\frac{m-1}{2}}}\exp\left(\frac{u}{2} - \frac{\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}}{2+2(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+v)u}\right) \\ & \times \frac{(-1)^\nu}{a^{\frac{\nu+1}{2}}}\sum_{k=0}^{\nu}\binom{\nu}{k}\frac{(-1)^{\nu+k}h_{\nu+k}(x_2) - h_{\nu+k}(x_1)}{(2\sqrt{a})^k} f_n(u)\,\mathrm{d}v\,\mathrm{d}u. \end{aligned} \tag{100}$$

In the simplest such case, $p = m-3$ (so $\nu = 0$), the sum collapses and the representation reduces further to

$$f_{\mathbf{q}_{\mathbf{L}}}(\mathbf{q}_{\mathbf{L}}) = \frac{\Gamma(n)\,|\lambda\mathbf{L}'\boldsymbol{\Sigma}^{-1}\mathbf{L}|^{-\frac{1}{2}}}{\Gamma(n-m+1)2^{\frac{m}{2}}\pi^{\frac{p+1}{2}}}\int_0^\infty\int_0^\infty \frac{uv^{\frac{1}{2}}}{[1+(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+v)u]^{\frac{m-1}{2}}}\exp\left(\frac{u}{2} - \frac{\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}}{2+2(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+v)u}\right)$$

$$\times \frac{\sqrt{\pi}}{2\sqrt{a}} \left[\operatorname{erf}\left(\frac{2a-b}{2\sqrt{a}}\right) + \operatorname{erf}\left(\frac{2a+b}{2\sqrt{a}}\right) \right] f_n(u)\, \mathrm{d}v\, \mathrm{d}u. \tag{101}$$

When $\hat{\boldsymbol{\mu}}_2 = \mathbf{0}_{m-p}$—equivalently $\delta = 0$, which includes the centred case $\boldsymbol{\mu} = \mathbf{0}_m$—the transformation above is inapplicable because $a \equiv 0$. In this case, however, the exponent in (76) no longer depends on x (both $\hat{\boldsymbol{\mu}}_2$ and $\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}$ vanish), and the inner integral is the elementary beta integral

$$\int_{-1}^{1} (1-x^2)^{\frac{m-p-3}{2}}\, \mathrm{d}x = \frac{\sqrt{\pi}\,\Gamma\left(\frac{m-p-1}{2}\right)}{\Gamma\left(\frac{m-p}{2}\right)}, \tag{102}$$

so that (76) collapses to a double integral for every *$p < m-1$, with no parity restriction and no division by a:*

$$f_{\mathbf{q_L}}(\mathbf{q_L}) = \frac{\Gamma(n)\, |\lambda \mathbf{L}'\boldsymbol{\Sigma}^{-1}\mathbf{L}|^{-\frac{1}{2}}}{\Gamma(n-m+1)\,\Gamma\left(\frac{m-p}{2}\right) 2^{\frac{m}{2}} \pi^{\frac{p}{2}}} \int_0^\infty \int_0^\infty \frac{u\, v^{\frac{m-p-2}{2}}}{[1+(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+v)u]^{\frac{m-1}{2}}} e^{-\frac{(\hat{\mathbf{q}}_1'\hat{\mathbf{q}}_1+v)u^2}{2}} f_n(u)\, \mathrm{d}v\, \mathrm{d}u. \tag{103}$$

For a single linear combination, the general representation of Theorem 2(iii) also becomes fully explicit under proportional covariance matrices when $n-m$ is even, as shown in the following remark.

Remark 2. *Suppose $\boldsymbol{\Sigma}_w = \boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}_z = \lambda\boldsymbol{\Sigma}$ for a positive scalar λ. We show that the conditional expectation appearing in (33) admits an explicit expression whenever $n-m$ is even. Write*

$$\boldsymbol{\alpha}'\mathbf{W}^{-1}\mathbf{z} = \tilde{\boldsymbol{\alpha}}'\tilde{\mathbf{W}}^{-1}\tilde{\mathbf{z}}, \tag{104}$$

where $\tilde{\mathbf{W}} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{W}\boldsymbol{\Sigma}^{-\frac{1}{2}} \sim \mathcal{W}_m(n, \mathbf{I}_m)$, $\tilde{\boldsymbol{\alpha}} = \sqrt{\lambda}\,\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\alpha}$, and $\tilde{\mathbf{z}} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{z}/\sqrt{\lambda} \sim \mathcal{N}_m(\tilde{\boldsymbol{\mu}}, \mathbf{I}_m)$ with $\tilde{\boldsymbol{\mu}} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\mu}/\sqrt{\lambda}$. We now apply Theorem 2(iii) to the triple $(\tilde{\mathbf{W}}, \tilde{\mathbf{z}}, \tilde{\boldsymbol{\alpha}})$, for which both covariance matrices equal $\mathbf{I}_m$. In (33), the constant then involves $(\tilde{\boldsymbol{\alpha}}'\tilde{\boldsymbol{\alpha}})^{\frac{1}{2}} = (\lambda\,\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha})^{\frac{1}{2}}$, while

$$\hat{u} = \tilde{\mathbf{z}}'\left(\mathbf{I}_m - \frac{\tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\alpha}}'}{\tilde{\boldsymbol{\alpha}}'\tilde{\boldsymbol{\alpha}}}\right)\tilde{\mathbf{z}}, \qquad \hat{y} = \frac{\tilde{\boldsymbol{\alpha}}'\tilde{\mathbf{z}} - v_1 q_{\boldsymbol{\alpha}}}{(\tilde{\boldsymbol{\alpha}}'\tilde{\boldsymbol{\alpha}})^{\frac{1}{2}}}. \tag{105}$$

Since $\tilde{\mathbf{z}} \sim \mathcal{N}_m(\tilde{\boldsymbol{\mu}}, \mathbf{I}_m)$, the projections $(\mathbf{I}_m - \tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\alpha}}'/\tilde{\boldsymbol{\alpha}}'\tilde{\boldsymbol{\alpha}})\tilde{\mathbf{z}}$ and $\tilde{\boldsymbol{\alpha}}'\tilde{\mathbf{z}}$ are independent, and using $\tilde{\boldsymbol{\alpha}}'\tilde{\boldsymbol{\mu}} = \boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/\lambda$, we obtain that, conditionally on v_1,

$$\hat{u} \sim \chi^2_{m-1}(\delta_u) \qquad \text{and} \qquad \hat{y}^2 \sim \chi^2_1(\delta_y) \tag{106}$$

are independent, where

$$\delta_u = \frac{1}{\lambda}\left[\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}\right], \qquad \delta_y = \frac{\left(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - v_1 q_{\boldsymbol{\alpha}}\right)^2}{\lambda\,\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}. \tag{107}$$

By Hillier et al. (2014, Theorem 4), an explicit expression for $\mathbb{E}\left[\hat{u}^r/(\hat{u}+\hat{y}^2)^s\right]$ is available when r is a nonnegative integer. Specifically, for $s = r + \frac{1}{2}$,

$$\mathbb{E}\left[\frac{\hat{u}^r}{(\hat{u}+\hat{y}^2)^{r+\frac{1}{2}}}\right] = \frac{r!}{\sqrt{2}} \sum_{l=0}^{r} \frac{\Gamma\left(\frac{m-1}{2}+l\right)}{2^l\, l!\, \Gamma\left(\frac{m}{2}+r+l\right)}\, {}_1F_1\left(r+\frac{1}{2}; \frac{m}{2}+r+l; -\frac{\delta}{2}\right) a_{r,l}, \tag{108}$$

where ${}_1F_1(\cdot;\cdot;\cdot)$ denotes the confluent hypergeometric function, $\delta = \delta_u + \delta_y$, and the coefficients $a_{r,l}$ are computed recursively as

$$a_{r,0} = \frac{\left(\frac{m-1}{2}\right)_r}{r!}, \qquad a_{r,l} = \delta_u \sum_{j=1}^{r-l+1} a_{r-j,l-1} \quad \text{for } l \geq 1. \tag{109}$$

This recursion immediately yields explicit expressions for any fixed r. For example, for $r = 1$,

$$\mathbb{E}\left[\frac{\hat{u}}{(\hat{u}+\hat{y}^2)^{\frac{3}{2}}}\right] = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{m+2}{2}\right)}\left[{}_1F_1\left(\frac{3}{2};\frac{m+2}{2};-\frac{\delta}{2}\right) + \frac{\delta_u}{m+2}\,{}_1F_1\left(\frac{3}{2};\frac{m+4}{2};-\frac{\delta}{2}\right)\right], \tag{110}$$

and for $r = 2$,

$$\mathbb{E}\left[\frac{\hat{u}^2}{(\hat{u}+\hat{y}^2)^{\frac{5}{2}}}\right] = \frac{\Gamma\left(\frac{m+3}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{m+4}{2}\right)}\left[{}_1F_1\left(\frac{5}{2};\frac{m+4}{2};-\frac{\delta}{2}\right) + \frac{2\delta_u}{m+4}\,{}_1F_1\left(\frac{5}{2};\frac{m+6}{2};-\frac{\delta}{2}\right)\right.$$
$$\left. + \frac{\delta_u^2}{(m+4)(m+6)}\,{}_1F_1\left(\frac{5}{2};\frac{m+8}{2};-\frac{\delta}{2}\right)\right]. \tag{111}$$

Numerical evaluation and validation

Every representation in this paper reduces to an integral of dimension at most six and, in the proportional case, at most three. These integrals are evaluated by Gauss–Legendre quadrature over the finite effective supports of the chi-square factors. Three points deserve mention. For the triple integral in Theorem 3(iii) the substitution $x = \sin\theta$ removes the endpoint singularity of $(1-x^2)^{(m-p-3)/2}$. In a neighbourhood of $q_{\boldsymbol{\alpha}} = 0$ the mixture form (94) is used in place of the parabolic cylinder expression, which is individually singular there; equivalently, the factor $e^{x^2/4}D_{-(n-m+3)/2}(x)$ is evaluated through the integral representation of the parabolic cylinder function, in which the divergent prefactor cancels analytically. In Proposition 1 the factor $(1+2\mathrm{i}\,\boldsymbol{\tau}'\mathbf{q_L})^{-(n-m+1+2p)/2}$ oscillates at frequency $\|\mathbf{q_L}\|$, so the number of quadrature nodes in $\boldsymbol{\tau}$ must be scaled with $\|\mathbf{q_L}\|$ to keep a fixed number of nodes per period. The chi-square densities and quantiles, together with the quadrature nodes, are computed directly, so no specialized software is required, and a complete 301-point density curve is obtained in a fraction of a second. Figure 1 was produced with this quadrature applied to Theorem 2(iii).

To validate the formulas, Table 1 compares the analytic density with the unbiased conditional Monte Carlo estimator

$$f_{\mathbf{q_L}}(\mathbf{q_L}) = \mathbb{E}_{\mathbf{W}}\left[\phi_p\left(\mathbf{q_L};\, \mathbf{L}'\mathbf{W}^{-1}\boldsymbol{\mu},\, \mathbf{L}'\mathbf{W}^{-1}\boldsymbol{\Sigma}_z\mathbf{W}^{-1}\mathbf{L}\right)\right], \tag{112}$$

where $\phi_p(\cdot;\boldsymbol{\eta},\boldsymbol{\Omega})$ is the $\mathcal{N}_p(\boldsymbol{\eta},\boldsymbol{\Omega})$ density. Conditionally on $\mathbf{W}$ the vector $\mathbf{q_L}$ is normal, so the estimator involves no kernel smoothing; the expectation is estimated by averaging over 10^6 Wishart draws. Throughout, $\mathbf{L}$ consists of the first p columns of $\mathbf{I}_m$, so that $\mathbf{q_L}$ collects the first p components of $\mathbf{W}^{-1}\mathbf{z}$ and only the evaluation point $\mathbf{q}_\mathbf{L}'$ has to be reported. The comparison covers the general and the proportional settings, every representation recommended above for computation, values of p equal to 1, 2, $m-1$ and m, both boundary cases identified in Section 4—namely $q_{\boldsymbol{\alpha}} = 0$ and $\hat{\boldsymbol{\mu}}_2 = \mathbf{0}_{m-p}$, equivalently $\delta = 0$—the origin $\mathbf{q} = \mathbf{0}_m$, and a non-integer degrees-of-freedom value close to the boundary $n = m-1$. In every case the analytic value agrees with the simulation to within 1.3 Monte Carlo standard errors, and fine-grid numerical integration of the $p = 1$ density gives 1.000000 in both settings.

5 Concluding remarks

The product of an inverse Wishart matrix and an independent normal vector is a central object in many statistical applications. It plays a special role in discriminant analysis, where the coefficients of the discriminant function are given by the product of the inverse of the sample covariance matrix and a sample mean difference vector, so that their joint distribution is determined by the distribution of an inverse Wishart–normal product (Muirhead, 1982; Rencher

Table 1: Analytic density versus the unbiased conditional Monte Carlo estimator (10^6 Wishart draws per row). In every row $m = 5$ and $\mathbf{L}$ consists of the first p columns of $\mathbf{I}_5$, so that $\mathbf{q}_\mathbf{L}$ collects the first p components of $\mathbf{W}^{-1}\mathbf{z}$; the column $\mathbf{q}'_\mathbf{L}$ gives the point at which the density is evaluated. Unless indicated otherwise, $\boldsymbol{\mu} = [1,\ 2,\ 3,\ 4,\ 5]'$ and $n = 10$. Panel A uses $\boldsymbol{\Sigma}_w = 0.5\,\mathbf{I}_5 + 0.5\,\mathbf{1}_5\mathbf{1}'_5$ and $\boldsymbol{\Sigma}_z = 0.2\,\mathbf{I}_5 + 0.8\,\mathbf{1}_5\mathbf{1}'_5$; Panel B uses $\boldsymbol{\Sigma}_w = \boldsymbol{\Sigma} = 0.4\,\mathbf{I}_5 + 0.6\,\mathbf{1}_5\mathbf{1}'_5$ and $\boldsymbol{\Sigma}_z = \lambda\boldsymbol{\Sigma}$ with $\lambda = 1.3$. Densities are reported to six significant digits; the last column gives the absolute discrepancy in Monte Carlo standard errors.

Representation	p	$\mathbf{q}'_\mathbf{L}$	Analytic	Monte Carlo	$\lvert\text{diff}\rvert$/s.e.
Panel A: general covariance matrices					
Proposition 1	1	(-0.75)	0.484219	0.484220	0.00
Proposition 1	2	$(-0.75,\ -0.25)$	0.337072	0.337244	0.16
Theorem 1, marginalized	4	$(-0.75,\ -0.25,\ 0.25,\ 0.75)$	0.218680	0.216593	0.70
Theorem 1	5	$(-0.75,\ -0.25,\ 0.25,\ 0.75,\ 1.25)$	0.0458902	0.0471129	0.47
Theorem 1, $\boldsymbol{\mu} = \mathbf{0}_5$	5	$(0,\ 0,\ 0,\ 0,\ 0)$	698.954	697.642	1.29
Panel B: proportional covariance matrices					
Theorem 3(iv)	1	(-0.75)	0.485049	0.485237	0.39
Theorem 3(iv)	1	(0)	0.322859	0.323209	0.74
Theorem 3(iii)	2	$(-0.75,\ -0.25)$	0.324821	0.324176	1.12
Theorem 3(iii), $\boldsymbol{\mu} = \mathbf{0}_5$	2	$(-0.75,\ -0.25)$	0.0765057	0.0765273	0.17
Theorem 3(ii)	4	$(-0.75,\ -0.25,\ 0.25,\ 0.75)$	0.187040	0.187846	0.84
Corollary 1	5	$(-0.75,\ -0.25,\ 0.25,\ 0.75,\ 1.25)$	0.0305264	0.0309247	0.71
Theorem 3(iv), $n = m - 0.5$	1	(-0.75)	0.0218069	0.0218705	0.84

and Christensen, 2012). The same product arises when optimal portfolio weights are estimated from historical asset return data (see, e.g., Bodnar and Okhrin, 2011; Karlsson et al., 2021; Kan et al., 2024b).

This paper derives the exact density function of the product of an inverse Wishart matrix and an independent normal vector under weak assumptions on the data-generating model: in particular, the scale matrix of the Wishart distribution and the covariance matrix of the normal vector are not required to coincide. The density is obtained as a one-dimensional integral that can be evaluated easily and accurately. We also provide integral representations for the density of linear combinations of the product, together with substantially simplified formulas in two practically important cases: a single linear combination and proportional covariance matrices. Moreover, a characteristic-function argument reduces the joint density of any p linear combinations, under arbitrary covariance matrices, to an absolutely convergent $(p+1)$-dimensional integral with a Gaussian-decaying integrand. Combining this representation with the marginalization route and with the six-dimensional route based on the joint density of $(u, v, \hat{u})$, the density of any collection of linear combinations can be evaluated from an integral of dimension at most $\min(p+1,\ m-p+1,\ 6)$. For the important special case in which the two covariance matrices are proportional, we provide simplified formulas involving integrals of dimension at most three.

We expect these results to sharpen the finite-sample understanding of the stochastic behaviour of discriminant function coefficients and of estimated optimal portfolio weights. They also open the way to higher-order moments of these quantities and to a study of their tail behaviour.

Appendix: Auxiliary lemmas

Lemma 1. *Let* $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{I}_m)$ *with* $n > m-1$. *For an* m*-vector* $\mathbf{q} \neq \mathbf{0}_m$, *let* $\mathbf{Q} = [\tilde{\mathbf{q}},\ \mathbf{Q}_1]$ *be an* $m \times m$ *orthonormal matrix whose first column is* $\tilde{\mathbf{q}} = \mathbf{q}/(\mathbf{q}'\mathbf{q})^{\frac{1}{2}}$. *Then*

$$|\mathbf{W}| \overset{d}{=} \prod_{i=1}^{m} u_i, \tag{A.1}$$

$$\mathbf{q}'\mathbf{W}\mathbf{q} \overset{d}{=} (\mathbf{q}'\mathbf{q})u_1, \tag{A.2}$$

$$\mathbf{Q}_1'\mathbf{W}\mathbf{q} \overset{d}{=} \sqrt{\mathbf{q}'\mathbf{q}}\sqrt{u_1}\,\mathbf{x}, \tag{A.3}$$

where $u_i \sim \chi^2_{n-i+1}$ $(i = 1, \ldots, m)$ *and* $\mathbf{x} \sim \mathcal{N}_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$ *are mutually independent.*

In addition, for an m*-vector* $\boldsymbol{\mu}$ *with* $(\mathbf{I}_m - \tilde{\mathbf{q}}\tilde{\mathbf{q}}')\boldsymbol{\mu} \neq \mathbf{0}_m$, *if the first column of* $\mathbf{Q}_1$ *equals*

$$\boldsymbol{\xi} = \frac{(\mathbf{I}_m - \tilde{\mathbf{q}}\tilde{\mathbf{q}}')\boldsymbol{\mu}}{[\boldsymbol{\mu}'(\mathbf{I}_m - \tilde{\mathbf{q}}\tilde{\mathbf{q}}')\boldsymbol{\mu}]^{\frac{1}{2}}}, \tag{A.4}$$

then

$$\mathbf{q}'\mathbf{W}\boldsymbol{\mu} \overset{d}{=} (\mathbf{q}'\boldsymbol{\mu})u_1 + \sqrt{c}\sqrt{u_1}\,x_1, \tag{A.5}$$

where x_1 *is the first element of* $\mathbf{x}$ *and* $c = (\mathbf{q}'\mathbf{q})(\boldsymbol{\mu}'\boldsymbol{\mu}) - (\mathbf{q}'\boldsymbol{\mu})^2$.

Proof. Let

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \mathbf{Q}'\mathbf{W}\mathbf{Q} \sim \mathcal{W}_m(n, \mathbf{I}_m), \tag{A.6}$$

where $\mathbf{A}_{11}$ is a scalar. From Muirhead (1982, Theorem 3.2.10),

$$\mathbf{A}_{11} \equiv u_1 \sim \chi^2_n, \tag{A.7}$$

$$\mathbf{A}_{21}\mathbf{A}_{11}^{-\frac{1}{2}} \equiv \mathbf{x} \sim \mathcal{N}_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1}), \tag{A.8}$$

$$\mathbf{A}_{22\cdot1} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \sim \mathcal{W}_{m-1}(n-1, \mathbf{I}_{m-1}), \tag{A.9}$$

and u_1, $\mathbf{x}$, and $\mathbf{A}_{22\cdot1}$ are mutually independent. Hence

$$\mathbf{q}'\mathbf{W}\mathbf{q} = (\mathbf{q}'\mathbf{q})\mathbf{A}_{11} \overset{d}{=} (\mathbf{q}'\mathbf{q})u_1, \tag{A.10}$$

$$\mathbf{Q}_1'\mathbf{W}\mathbf{q} = (\mathbf{q}'\mathbf{q})^{\frac{1}{2}}\mathbf{A}_{21} = (\mathbf{q}'\mathbf{q})^{\frac{1}{2}}\mathbf{A}_{21}\mathbf{A}_{11}^{-\frac{1}{2}}\mathbf{A}_{11}^{\frac{1}{2}} \overset{d}{=} \sqrt{\mathbf{q}'\mathbf{q}}\sqrt{u_1}\,\mathbf{x}. \tag{A.11}$$

Moreover, $|\mathbf{W}| = |\mathbf{A}_{11}||\mathbf{A}_{22\cdot1}| = u_1|\mathbf{A}_{22\cdot1}|$, and by Muirhead (1982, Theorem 3.2.15),

$$|\mathbf{A}_{22\cdot1}| \overset{d}{=} \prod_{i=2}^{m} u_i, \tag{A.12}$$

where $u_i \sim \chi^2_{n+1-i}$ $(i = 2, \ldots, m)$ and $u_1, \ldots, u_m$ are mutually independent.

When the first column of $\mathbf{Q}_1$ is $\boldsymbol{\xi}$, we have

$$\mathbf{q}'\mathbf{W}\boldsymbol{\xi} = (\mathbf{q}'\mathbf{q})^{\frac{1}{2}}\,\tilde{\mathbf{q}}'\mathbf{W}\mathbf{Q}_1 \begin{bmatrix} 1 \\ \mathbf{0}_{m-2} \end{bmatrix} \overset{d}{=} (\mathbf{q}'\mathbf{q})^{\frac{1}{2}}\sqrt{u_1}\,x_1, \tag{A.13}$$

where x_1 is the first element of $\mathbf{x}$. Using

$$\mathbf{q}'\mathbf{W}\boldsymbol{\xi} = \frac{\mathbf{q}'\mathbf{W}(\mathbf{I}_m - \tilde{\mathbf{q}}\tilde{\mathbf{q}}')\boldsymbol{\mu}}{[\boldsymbol{\mu}'\boldsymbol{\mu} - (\mathbf{q}'\boldsymbol{\mu})^2/(\mathbf{q}'\mathbf{q})]^{\frac{1}{2}}} = \frac{\mathbf{q}'\mathbf{W}\boldsymbol{\mu} - (\tilde{\mathbf{q}}'\mathbf{W}\tilde{\mathbf{q}})\,\mathbf{q}'\boldsymbol{\mu}}{(\mathbf{q}'\mathbf{q})^{-\frac{1}{2}}\sqrt{c}}, \tag{A.14}$$

we obtain

$$\mathbf{q}'\mathbf{W}\boldsymbol{\mu} = (\tilde{\mathbf{q}}'\mathbf{W}\tilde{\mathbf{q}})\,\mathbf{q}'\boldsymbol{\mu} + (\mathbf{q}'\mathbf{q})^{-\frac{1}{2}}\sqrt{c}\,\mathbf{q}'\mathbf{W}\boldsymbol{\xi} \overset{d}{=} (\mathbf{q}'\boldsymbol{\mu})u_1 + \sqrt{c}\sqrt{u_1}\,x_1. \tag{A.15}$$

□

Lemma 2. *Suppose* $\mathbf{y} \sim \mathcal{N}_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$*. Then, for any* $(m-1)$*-dimensional vector* $\mathbf{b}_1$ *and any* $(m-1) \times (m-1)$ *positive semidefinite matrix* $\mathbf{B}$*,*

$$\mathbb{E}\left[\exp\left(\mathbf{b}_1'\mathbf{y} - \frac{\mathbf{y}'\mathbf{B}\mathbf{y}}{2}\right)\right] = |\mathbf{I}_{m-1} + \mathbf{B}|^{-\frac{1}{2}} \exp\left(\frac{\mathbf{b}_1'(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1}{2}\right). \tag{A.16}$$

Proof. Using the density of $\mathbf{y}$,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\mathbf{b}_1'\mathbf{y} - \frac{\mathbf{y}'\mathbf{B}\mathbf{y}}{2}\right)\right] \\
&= \frac{1}{(2\pi)^{\frac{m-1}{2}}} \int_{\mathbb{R}^{m-1}} \exp\left(\mathbf{b}_1'\mathbf{y} - \frac{\mathbf{y}'\mathbf{B}\mathbf{y}}{2}\right) \exp\left(-\frac{\mathbf{y}'\mathbf{y}}{2}\right) \mathrm{d}\mathbf{y} \\
&= |\mathbf{I}_{m-1} + \mathbf{B}|^{-\frac{1}{2}} \exp\left(\frac{\mathbf{b}_1'(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1}{2}\right) \\
&\quad \times \int_{\mathbb{R}^{m-1}} \frac{|\mathbf{I}_{m-1}+\mathbf{B}|^{\frac{1}{2}}}{(2\pi)^{\frac{m-1}{2}}} \exp\left(-\frac{\left(\mathbf{y} - (\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1\right)'(\mathbf{I}_{m-1}+\mathbf{B})\left(\mathbf{y} - (\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1\right)}{2}\right) \mathrm{d}\mathbf{y} \\
&= |\mathbf{I}_{m-1} + \mathbf{B}|^{-\frac{1}{2}} \exp\left(\frac{\mathbf{b}_1'(\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1}{2}\right), \qquad \text{(A.17)}
\end{aligned}$$

where the last equality holds because the remaining integrand is the density function of a $\mathcal{N}_{m-1}\left((\mathbf{I}_{m-1}+\mathbf{B})^{-1}\mathbf{b}_1, (\mathbf{I}_{m-1}+\mathbf{B})^{-1}\right)$ random vector. □

The next lemma is the inverse Wishart counterpart of Lemma 1: it provides a joint stochastic representation for the quadratic form $\tilde{\mathbf{q}}'\mathbf{W}^{-1}\tilde{\mathbf{q}}$ and the vector $\mathbf{Q}_1'\mathbf{W}^{-1}\tilde{\mathbf{q}}$.

Lemma 3. *Let* $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{I}_m)$ *with* $m \geq 2$ *and* $n > m-1$*. For an* m*-vector* $\mathbf{q} \neq \mathbf{0}_m$*, let* $\tilde{\mathbf{q}} = \mathbf{q}/(\mathbf{q}'\mathbf{q})^{\frac{1}{2}}$ *and let* $\mathbf{Q}_1$ *be an* $m \times (m-1)$ *orthonormal matrix whose columns are orthogonal to* $\mathbf{q}$*. Then, jointly,*

$$\tilde{\mathbf{q}}'\mathbf{W}^{-1}\tilde{\mathbf{q}} \overset{d}{=} \frac{1}{v_1}, \qquad \mathbf{Q}_1'\mathbf{W}^{-1}\tilde{\mathbf{q}} \overset{d}{=} \frac{\mathbf{x}}{v_1\sqrt{v_2}}, \tag{A.18}$$

where $\mathbf{x} \sim \mathcal{N}_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$*,* $v_1 \sim \chi^2_{n-m+1}$*, and* $v_2 \sim \chi^2_{n-m+2}$ *are mutually independent.*

Proof. Let $\mathbf{Q} = [\tilde{\mathbf{q}}, \ \mathbf{Q}_1]$, which is an $m \times m$ orthonormal matrix, and set

$$\mathbf{A} = \begin{bmatrix} A_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \mathbf{Q}'\mathbf{W}\mathbf{Q} \sim \mathcal{W}_m(n, \mathbf{I}_m), \tag{A.19}$$

where A_{11} is a scalar. Since $\mathbf{Q}\mathbf{Q}' = \mathbf{I}_m$, we have $\mathbf{Q}'\mathbf{W}^{-1}\mathbf{Q} = (\mathbf{Q}'\mathbf{W}\mathbf{Q})^{-1} = \mathbf{A}^{-1}$, and the partitioned inverse formula gives

$$\tilde{\mathbf{q}}'\mathbf{W}^{-1}\tilde{\mathbf{q}} = [\mathbf{A}^{-1}]_{11} = A_{11\cdot 2}^{-1}, \qquad \mathbf{Q}_1'\mathbf{W}^{-1}\tilde{\mathbf{q}} = [\mathbf{A}^{-1}]_{21} = -\mathbf{A}_{22}^{-\frac{1}{2}}\left(\mathbf{A}_{22}^{-\frac{1}{2}}\mathbf{A}_{21}\right)A_{11\cdot 2}^{-1}, \tag{A.20}$$

where $A_{11\cdot 2} = A_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$. Applying Muirhead (1982, Theorem 3.2.10) to $\mathbf{A}$, with the first block of size one, yields

$$A_{11\cdot 2} \equiv v_1 \sim \chi^2_{n-m+1}, \quad \text{independent of } (\mathbf{A}_{21}, \mathbf{A}_{22}), \tag{A.21}$$

$$\mathbf{A}_{21} \mid \mathbf{A}_{22} \sim \mathcal{N}_{m-1}(\mathbf{0}_{m-1}, \mathbf{A}_{22}), \tag{A.22}$$

$$\mathbf{A}_{22} \sim \mathcal{W}_{m-1}(n, \mathbf{I}_{m-1}). \tag{A.23}$$

The second line implies that $\mathbf{x}_0 = \mathbf{A}_{22}^{-\frac{1}{2}}\mathbf{A}_{21} \sim \mathcal{N}_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$ unconditionally and that $\mathbf{x}_0$ is independent of $\mathbf{A}_{22}$. Moreover, $\mathbf{A}_{22}^{-\frac{1}{2}}\mathbf{x}_0$ is a function of $(\mathbf{A}_{21}, \mathbf{A}_{22})$ and is therefore independent of v_1.

It remains to show that

$$\mathbf{A}_{22}^{-\frac{1}{2}}\mathbf{x}_0 \overset{d}{=} \frac{\mathbf{x}}{\sqrt{v_2}}, \tag{A.24}$$

where $\mathbf{x} \sim \mathcal{N}_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$ and $v_2 \sim \chi^2_{n-m+2}$ are independent. First, $\mathbf{A}_{22}^{-\frac{1}{2}}\mathbf{x}_0$ is spherically symmetric: for any fixed $(m-1) \times (m-1)$ orthogonal matrix $\mathbf{O}$,

$$\mathbf{O}\mathbf{A}_{22}^{-\frac{1}{2}}\mathbf{x}_0 = \left(\mathbf{O}\mathbf{A}_{22}\mathbf{O}'\right)^{-\frac{1}{2}}(\mathbf{O}\mathbf{x}_0) \overset{d}{=} \mathbf{A}_{22}^{-\frac{1}{2}}\mathbf{x}_0, \tag{A.25}$$

where the first equality holds because the symmetric square root commutes with orthogonal conjugation, and the distributional equality holds because the orthogonal invariance of the Wishart and standard normal distributions gives $(\mathbf{O}\mathbf{A}_{22}\mathbf{O}', \mathbf{O}\mathbf{x}_0) \overset{d}{=} (\mathbf{A}_{22}, \mathbf{x}_0)$. Second, consider the squared norm $\mathbf{x}_0'\mathbf{A}_{22}^{-1}\mathbf{x}_0$. By Muirhead (1982, Theorem 3.2.12), for $\mathbf{W}_0 \sim \mathcal{W}_k(n, \mathbf{I}_k)$ and any fixed $\mathbf{a} \neq \mathbf{0}_k$ we have $(\mathbf{a}'\mathbf{a})/(\mathbf{a}'\mathbf{W}_0^{-1}\mathbf{a}) \sim \chi^2_{n-k+1}$. Applying this with $k = m-1$ conditionally on $\mathbf{x}_0$ shows that

$$v_2 = \left.\frac{\mathbf{x}_0'\mathbf{x}_0}{\mathbf{x}_0'\mathbf{A}_{22}^{-1}\mathbf{x}_0}\right| \mathbf{x}_0 \sim \chi^2_{n-m+2}, \tag{A.26}$$

and since this conditional distribution does not depend on $\mathbf{x}_0$, the random variable v_2 is independent of $\mathbf{x}_0$. Hence $\mathbf{x}_0'\mathbf{A}_{22}^{-1}\mathbf{x}_0 = \mathbf{x}_0'\mathbf{x}_0/v_2$, where $\mathbf{x}_0'\mathbf{x}_0 \sim \chi^2_{m-1}$ and $v_2 \sim \chi^2_{n-m+2}$ are independent. The vector $\mathbf{x}/\sqrt{v_2}$ on the right-hand side of (A.24) is likewise spherically symmetric, and its squared norm $\mathbf{x}'\mathbf{x}/v_2$ has the same distribution. Since a spherically symmetric distribution is uniquely determined by the distribution of its Euclidean norm, (A.24) follows.

Combining these results, and using $-\mathbf{x} \overset{d}{=} \mathbf{x}$ to absorb the sign in (A.20), we obtain the joint representation

$$\left(A_{11\cdot 2},\ \mathbf{A}_{22}^{-\frac{1}{2}}\mathbf{x}_0\right) \overset{d}{=} \left(v_1,\ \frac{\mathbf{x}}{\sqrt{v_2}}\right) \tag{A.27}$$

with v_1, v_2, and $\mathbf{x}$ mutually independent, which together with (A.20) delivers (A.18). □

The next lemma provides a stochastic representation for a single linear combination $q_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{W}^{-1}\mathbf{z}$. It is not required for the proofs of the main results, but it underlies an efficient simulation scheme for $q_{\boldsymbol{\alpha}}$ and is of independent interest.

Lemma 4. *Let $\mathbf{W} \sim \mathcal{W}_m(n, \boldsymbol{\Sigma}_w)$ and $\mathbf{z} \sim \mathcal{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}_z)$ be independent, and let $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\boldsymbol{\alpha}$ for an m-dimensional vector $\boldsymbol{\alpha}$. Consider the spectral decomposition*

$$\mathbf{P}\mathbf{D}\mathbf{P}' = \boldsymbol{\Sigma}_z^{\frac{1}{2}}\boldsymbol{\Sigma}_w^{-\frac{1}{2}}\left(\mathbf{I}_m - \frac{\tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\alpha}}'}{\tilde{\boldsymbol{\alpha}}'\tilde{\boldsymbol{\alpha}}}\right)\boldsymbol{\Sigma}_w^{-\frac{1}{2}}\boldsymbol{\Sigma}_z^{\frac{1}{2}}, \tag{A.28}$$

where $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_m)$ contains the eigenvalues in descending order (so that $d_m = 0$), and $\mathbf{P}$ contains the corresponding eigenvectors. In addition, let

$$\mathbf{y} = \mathbf{P}'\boldsymbol{\Sigma}_z^{-\frac{1}{2}}\mathbf{z} \sim \mathcal{N}_m(\boldsymbol{\mu}_y, \mathbf{I}_m), \qquad \boldsymbol{\mu}_y = \mathbf{P}'\boldsymbol{\Sigma}_z^{-\frac{1}{2}}\boldsymbol{\mu}, \tag{A.29}$$

and $\hat{\boldsymbol{\alpha}} = \mathbf{P}'\boldsymbol{\Sigma}_z^{\frac{1}{2}}\boldsymbol{\Sigma}_w^{-1}\boldsymbol{\alpha}$. Then $q_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{W}^{-1}\mathbf{z}$ admits the stochastic representation

$$q_{\boldsymbol{\alpha}} \overset{d}{=} \frac{\hat{\alpha}_m \mu_{y,m} + v}{v_1} + \sigma w, \tag{A.30}$$

where

$$\sigma^2 = \frac{\hat{\alpha}_m^2}{v_1^2} + \frac{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w^{-1}\boldsymbol{\alpha})u}{v_1^2 v_2}, \tag{A.31}$$

$w \sim \mathcal{N}(0,1)$, $v_1 \sim \chi^2_{n-m+1}$, $v_2 \sim \chi^2_{n-m+2}$, $u = \sum_{i=1}^{m-1} d_i y_i^2$, and $v = \sum_{i=1}^{m-1} \hat{\alpha}_i y_i$; here w, v_1, v_2, and $(y_1, \dots, y_{m-1})$ are mutually independent.

Proof. Let $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\boldsymbol{\alpha}$ and $\tilde{\mathbf{z}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\mathbf{z}$, so that

$$\boldsymbol{\alpha}'\mathbf{W}^{-1}\mathbf{z} = \tilde{\boldsymbol{\alpha}}'\tilde{\mathbf{W}}^{-1}\tilde{\mathbf{z}}, \tag{A.32}$$

where $\tilde{\mathbf{W}} = \boldsymbol{\Sigma}_w^{-\frac{1}{2}}\mathbf{W}\boldsymbol{\Sigma}_w^{-\frac{1}{2}} \sim \mathcal{W}_m(n, \mathbf{I}_m)$. Let $\bar{\mathbf{z}} = \tilde{\mathbf{z}}/(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{\frac{1}{2}}$ and let $\mathbf{Q}$ be an $m \times (m-1)$ orthonormal matrix whose columns are orthogonal to $\tilde{\mathbf{z}}$. Since $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{z}}$ are independent, applying Lemma 3 conditionally on $\tilde{\mathbf{z}}$ (with $\mathbf{q} = \tilde{\mathbf{z}}$) gives

$$\bar{\mathbf{z}}'\tilde{\mathbf{W}}^{-1}\bar{\mathbf{z}} \overset{d}{=} \frac{1}{v_1}, \qquad \mathbf{Q}'\tilde{\mathbf{W}}^{-1}\bar{\mathbf{z}} \overset{d}{=} \frac{\mathbf{x}}{v_1\sqrt{v_2}}, \tag{A.33}$$

where $\mathbf{x} \sim \mathcal{N}_{m-1}(\mathbf{0}_{m-1}, \mathbf{I}_{m-1})$, $v_1 \sim \chi^2_{n-m+1}$, and $v_2 \sim \chi^2_{n-m+2}$ are mutually independent; since their joint distribution does not depend on the conditioning value, they are also independent of $\tilde{\mathbf{z}}$. It follows that

$$\tilde{\boldsymbol{\alpha}}'\tilde{\mathbf{W}}^{-1}\tilde{\mathbf{z}} = \tilde{\boldsymbol{\alpha}}'[\bar{\mathbf{z}},\ \mathbf{Q}][\bar{\mathbf{z}},\ \mathbf{Q}]'\tilde{\mathbf{W}}^{-1}\tilde{\mathbf{z}} = [\tilde{\boldsymbol{\alpha}}'\tilde{\mathbf{z}},\ (\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{\frac{1}{2}}\tilde{\boldsymbol{\alpha}}'\mathbf{Q}] \begin{bmatrix} \bar{\mathbf{z}}'\tilde{\mathbf{W}}^{-1}\bar{\mathbf{z}} \\ \mathbf{Q}'\tilde{\mathbf{W}}^{-1}\bar{\mathbf{z}} \end{bmatrix} \overset{d}{=} \frac{\tilde{\boldsymbol{\alpha}}'\tilde{\mathbf{z}}}{v_1} + \frac{(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{\frac{1}{2}}}{v_1\sqrt{v_2}}\tilde{\boldsymbol{\alpha}}'\mathbf{Q}\mathbf{x}. \tag{A.34}$$

Conditional on $\tilde{\mathbf{z}}$, and using $\mathbf{Q}\mathbf{Q}' = \mathbf{I}_m - \tilde{\mathbf{z}}(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{-1}\tilde{\mathbf{z}}'$,

$$(\tilde{\mathbf{z}}'\tilde{\mathbf{z}})^{\frac{1}{2}}\,\tilde{\boldsymbol{\alpha}}'\mathbf{Q}\mathbf{x} \,\big|\, \tilde{\mathbf{z}} \sim \mathcal{N}\big(0, (\tilde{\boldsymbol{\alpha}}'\tilde{\boldsymbol{\alpha}})u\big), \tag{A.35}$$

because

$$\tilde{\mathbf{z}}'\left(\mathbf{I}_m - \frac{\tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\alpha}}'}{\tilde{\boldsymbol{\alpha}}'\tilde{\boldsymbol{\alpha}}}\right)\tilde{\mathbf{z}} = \tilde{\mathbf{z}}'\boldsymbol{\Sigma}_w^{\frac{1}{2}}\boldsymbol{\Sigma}_z^{-\frac{1}{2}}\mathbf{P}\mathbf{P}'\boldsymbol{\Sigma}_z^{\frac{1}{2}}\boldsymbol{\Sigma}_w^{-\frac{1}{2}}\left(\mathbf{I}_m - \frac{\tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\alpha}}'}{\tilde{\boldsymbol{\alpha}}'\tilde{\boldsymbol{\alpha}}}\right)\boldsymbol{\Sigma}_w^{-\frac{1}{2}}\boldsymbol{\Sigma}_z^{\frac{1}{2}}\mathbf{P}\mathbf{P}'\boldsymbol{\Sigma}_z^{-\frac{1}{2}}\boldsymbol{\Sigma}_w^{\frac{1}{2}}\tilde{\mathbf{z}} = \mathbf{y}'\mathbf{D}\mathbf{y} = u. \tag{A.36}$$

In addition,

$$\tilde{\boldsymbol{\alpha}}'\tilde{\mathbf{z}} = \tilde{\boldsymbol{\alpha}}'\boldsymbol{\Sigma}_w^{-\frac{1}{2}}\boldsymbol{\Sigma}_z^{\frac{1}{2}}\mathbf{P}\mathbf{P}'\boldsymbol{\Sigma}_z^{-\frac{1}{2}}\boldsymbol{\Sigma}_w^{\frac{1}{2}}\tilde{\mathbf{z}} = \hat{\boldsymbol{\alpha}}'\mathbf{y} = \hat{\alpha}_m y_m + v. \tag{A.37}$$

Conditional on $(y_1, \dots, y_{m-1})$, v_1, and v_2, the term $\hat{\alpha}_m y_m$ contributes $\mathcal{N}(\hat{\alpha}_m\mu_{y,m}, \hat{\alpha}_m^2)$, so

$$\boldsymbol{\alpha}'\mathbf{W}^{-1}\mathbf{z} \,\big|\, y_1, \dots, y_{m-1}, v_1, v_2 \sim \mathcal{N}\left(\frac{\hat{\alpha}_m\mu_{y,m} + v}{v_1},\ \frac{\hat{\alpha}_m^2 + \frac{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}_w^{-1}\boldsymbol{\alpha})u}{v_2}}{v_1^2}\right), \tag{A.38}$$

which is the stated representation. □

Lemma 5. *Suppose $\mathbf{x} \sim \mathcal{N}_m(\boldsymbol{\mu}, \mathbf{I}_m)$. Let $\mathbf{D} = \mathrm{diag}(d_1, \dots, d_m)$ be a diagonal matrix with nonnegative elements, and define*

$$U = \mathbf{x}'\mathbf{D}\mathbf{x}, \tag{A.39}$$

$$V = \mathbf{x}'\mathbf{D}\mathbf{x} + (\mathbf{a}'\mathbf{x} - b)^2 = \mathbf{x}'(\mathbf{D} + \mathbf{a}\mathbf{a}')\mathbf{x} - 2b\mathbf{a}'\mathbf{x} + b^2, \tag{A.40}$$

where $\mathbf{D} + \mathbf{a}\mathbf{a}'$ is assumed to be of full rank. When r is a nonnegative integer,

$$\mu_s^r \equiv \mathbb{E}\left[\frac{U^r}{V^s}\right] = \frac{1}{\Gamma(s)}\int_0^\infty t_2^{s-1}\psi(0, -t_2)\,\mathbb{E}[(\mathbf{w}'\mathbf{R}\mathbf{w})^r]\,\mathrm{d}t_2, \tag{A.41}$$

where $\mathbf{R} = \mathbf{T}'\mathbf{D}\mathbf{T}$*,* $\mathbf{w} \sim \mathcal{N}_m(\tilde{\boldsymbol{\mu}}, \mathbf{I}_m)$ *with* $\tilde{\boldsymbol{\mu}} = \mathbf{T}'(\boldsymbol{\mu} + 2t_2 b\mathbf{a})$*, and* $\mathbf{T}$ *is a lower triangular matrix such that* $\mathbf{T}\mathbf{T}' = (\mathbf{I}_m + 2t_2(\mathbf{D} + \mathbf{a}\mathbf{a}'))^{-1}$*. When r is not an integer,*

$$\mu_s^r \equiv \mathbb{E}\left[\frac{U^r}{V^s}\right] = \frac{1}{\Gamma(\langle r\rangle)\Gamma(s)} \int_0^\infty \int_0^\infty t_1^{\langle r\rangle - 1} t_2^{s-1} \psi(-t_1, -t_2)\, \mathbb{E}[(\mathbf{w}'\mathbf{R}\mathbf{w})^{\lceil r \rceil}]\, \mathrm{d}t_1\, \mathrm{d}t_2, \tag{A.42}$$

where $\lceil r \rceil$ *denotes the smallest integer greater than or equal to r,* $\langle r \rangle = \lceil r \rceil - r \in (0,1)$*, and* $\mathbf{T}$ *is now a lower triangular matrix such that* $\mathbf{T}\mathbf{T}' = (\mathbf{I}_m + 2t_1\mathbf{D} + 2t_2(\mathbf{D} + \mathbf{a}\mathbf{a}'))^{-1}$*. Here* $\psi(t_1, t_2)$ *denotes the joint moment-generating function of* (U, V)*, which is given by (Mathai and Provost, 1992, Theorem 3.2c.1)*

$$\psi(t_1, t_2) = |\mathbf{I}_m - 2t_1\mathbf{D} - 2t_2(\mathbf{D} + \mathbf{a}\mathbf{a}')|^{-\frac{1}{2}} \exp\left(-\frac{\boldsymbol{\mu}'\boldsymbol{\mu}}{2} + t_2 b^2\right)$$
$$\times \exp\left(\frac{1}{2}(\boldsymbol{\mu} - 2t_2 b\mathbf{a})'(\mathbf{I}_m - 2t_1\mathbf{D} - 2t_2(\mathbf{D} + \mathbf{a}\mathbf{a}'))^{-1}(\boldsymbol{\mu} - 2t_2 b\mathbf{a})\right). \tag{A.43}$$

Moreover, $\mathbb{E}[(\mathbf{w}'\mathbf{R}\mathbf{w})^k]$ *for a nonnegative integer k can be computed by standard recursions for moments of quadratic forms in normal random vectors.*

Proof. The proof parallels that of Bao and Kan (2013, Proposition 2) and is omitted. □

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